

Conformal Ward and BPZ Identities for Liouville quantum field theory

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Abstract

In this work, we continue the constructive probabilistic approach to the Liouville Quantum Field theory (LQFT) started in [8]. We give a rigorous construction of the stress energy tensor in LQFT and prove the validity of the conformal Ward identities. Within this framework, we also derive the Belavin-Polyakov-Zamolodchikov (BPZ) differential equations of order 2 for the associated degenerate fields of the theory. As an application, we give an explicit formula for the 4 point correlation function with a degenerate field insertion leading to a proof of a non trivial functional relation on the 3 point structure constant derived earlier by Teschner.

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1 Introduction

Liouville Quantum Field theory (LQFT) is an important family of conformal field theories (CFT) indexed by one parameter $\gamma \in]0, 2[$ ^{1 2}. First, it is a building block of Polyakov’s formulation of Liouville quantum gravity (LQG) [23], a natural random version of Riemannian geometry; second, it is in some sense the simplest class of CFTs expected to carry a continuous spectrum of highest weight representations of the Virasoro Algebra (see e.g. Chapter 3 in [24]).

In a previous work [8], F. David and the authors of the present paper started a rigorous probabilistic approach to LQFT on the Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ equipped with a smooth conformal metric $g = g(z)dz^2$. More precisely, we introduce $Q = \frac{\gamma}{2} + \frac{2}{\gamma}$ and a parameter $\mu > 0$. For $N \geq 3$ let z_1, \dots, z_N be distinct points in \mathbb{C} and let $\alpha_1, \dots, \alpha_N$ be real numbers satisfying the so-called *Seiberg bounds*

$$\alpha_l < Q \quad \forall l, \quad \sum_i \alpha_i > 2Q. \quad (1.1)$$

The authors of [8] gave a rigorous definition for any smooth function F on the Sobolev space $H^{-1}(\hat{\mathbb{C}})$ of the following formal path integral

$$\langle \prod_l V_{\alpha_l}(z_l) F(X) \rangle := \int \prod_l V_{\alpha_l}(z_l) F(X) e^{-S(X,g)} DX \quad (1.2)$$

where the $V_{\alpha_l}(z_l) = e^{\alpha_l X(z_l)}$ are the so-called vertex operators (in the physics terminology) and $S(X, g)$ is the Liouville action functional

$$S(X, g) := \frac{1}{4\pi} \int_{\mathbb{C}} (|\partial^g X|^2 + Q R_g X + 4\pi \mu e^{\gamma X}) d\lambda_g, \quad (1.3)$$

where ∂^g , R_g and λ_g respectively stand for the gradient, Ricci scalar curvature and volume measure in the metric g . The parameter $\mu > 0$ is the analog of a “cosmological constant” in two dimensional gravity and the choice $Q = \frac{\gamma}{2} + \frac{2}{\gamma}$ ensures conformal invariance properties of the theory.

Making sense of (1.2) requires taking the limit in a regularization and renormalization procedure and then yields a CFT with central charge $c_L = 1 + 6Q^2$ (hence the central charge of LQFT ranges continuously

¹In the physics literature $b = \frac{\gamma}{2}$ is often used.

²Extensions to $\gamma \in \mathbb{C}$ have also been considered but we will not study these cases here.

in $[25, \infty[$; we will recall this in the next section. Furthermore, dependence on the parameter μ is simple and is covered by the KPZ scaling law (after Khniznik-Polyakov-Zamolodchikov). Thus effectively the theory depends only on one parameter, γ .

In the language of CFT, the vertex operators V_α are primary fields whose correlation functions $\langle \prod_l V_{\alpha_l}(z_l) \rangle$ exhibit conformal invariance properties; indeed, it is shown in [8] that for ψ a Möbius transform we have the following transformation property:

$$\langle \prod_l V_{\alpha_l}(\psi(z_l)) \rangle = \prod_l |\psi'(z_l)|^{-2\Delta_{\alpha_l}} \langle \prod_l V_{\alpha_l}(z_l) \rangle \quad (1.4)$$

where the exponents $\Delta_{\alpha_l} = \frac{\alpha_l}{2}(Q - \frac{\alpha_l}{2})$ are called conformal weights. The goal of a CFT is to construct all the primary fields and compute the associated correlation functions; we believe this paper is a step towards that goal.

Another important building block of a CFT is the stress-energy tensor (SET) which encodes the variations of the theory with respect to the background metric g . More specifically, suppose the correlation functions are defined in an arbitrary smooth Riemannian metric and write the inverse metric in complex coordinates as

$$g^{-1} = g^{zz} \partial_z \otimes \partial_z + \frac{1}{2} g^{z\bar{z}} (\partial_z \otimes \partial_{\bar{z}} + \partial_{\bar{z}} \otimes \partial_z) + g^{\bar{z}\bar{z}} \partial_{\bar{z}} \otimes \partial_{\bar{z}}.$$

Let f be a smooth function with support in $\mathbb{C} \setminus \cup_i z_i$ and $g_\epsilon^{-1} = g^{-1} + \epsilon f \partial_z \otimes \partial_z$. Then (a component of) the stress tensor $T(z)$ is defined by the following formula in the physics literature

$$\frac{d}{d\epsilon} \Big|_{\epsilon=0} \langle \prod_l V_{\alpha_l}(z_l) \rangle_{g_\epsilon} := \int f(z) \langle T(z) \prod_l V_{\alpha_l}(z_l) \rangle_g dz. \quad (1.5)$$

A simple formal computation then yields the following heuristic formula

$$T(z) = -(\partial_z \phi)^2(z) + Q \partial_z^2 \phi(z). \quad (1.6)$$

where ϕ is the *Liouville field*

$$\phi = X + \frac{1}{2} Q \ln g \quad (1.7)$$

and as we will see (1.6) requires a regularization and renormalization procedure to be defined properly.

The importance of the SET in CFT lies in the fact that it can be used to describe the *local conformal symmetries* in CFT. In the axiomatic approach to CFT (see e.g. Gawedzki's lecture notes [17]) the starting point for this are explicit relations satisfied by the correlation functions with T -insertion: the *Conformal Ward Identities*. The first Ward identity controls the singularity as the argument of T gets close to one of the V_α :

$$\langle T(z) \prod_l V_{\alpha_l}(z_l) \rangle = \sum_k \frac{\Delta_{\alpha_k}}{(z - z_k)^2} \langle \prod_l V_{\alpha_l}(z_l) \rangle - \sum_k \frac{1}{z - z_k} \partial_{z_k} \langle \prod_l V_{\alpha_l}(z_l) \rangle \quad (1.8)$$

Note in particular that the T insertion is holomorphic. The second identity controls the singularity when two T -insertions come close and is symbolically written as

$$T(z)T(z') = \frac{\frac{1}{2}c_L}{(z - z')^4} + \frac{2}{(z - z')^2} T(z') + \frac{1}{z - z'} \partial_{z'} T(z') + \dots \quad (1.9)$$

where the dots refer to terms that are bounded as $z \rightarrow z'$.

The goal of this paper is to define the stress tensor rigorously and prove the first Ward identity (1.8). More precisely, we will define $T(z)$ by a regularization procedure associated to definition (1.6) rather than try to make a rigorous definition out of (1.5). In fact, showing that the correlation functions are differentiable with respect to the background metric (or inverse metric) is non trivial hence we leave this alternative construction of the stress tensor as an open problem. We will discuss the second identity (1.9) in a separate publication.

Since T is a holomorphic observable, it contains important information on the theory, which we will in fact exploit to deduce from our framework the so-called BPZ (after Belavin-Polyakov-Zamolodchikov [3]) equations for the vertex operator $V_{-\frac{\gamma}{2}}$ (in the language of CFT, $V_{-\frac{\gamma}{2}}$ is called a degenerate field). More precisely, setting $F(z, z_1, \dots, z_N) = \langle V_{-\frac{\gamma}{2}}(z) \prod_l V_{\alpha_l}(z_l) \rangle$, we prove

$$\frac{4}{\gamma^2} \partial_z^2 F + \sum_k \frac{\Delta_{\alpha_k}}{(z - z_k)^2} F + \sum_k \frac{1}{z - z_k} \partial_{z_k} F = 0, \quad \textbf{BPZ equations.}$$

As an application of the BPZ equation, we will recover an explicit formula found earlier in the physics literature for the 4 point correlation function $\langle V_{-\frac{\gamma}{2}}(z) \prod_{l=1}^3 V_{\alpha_l}(z_l) \rangle$. Following what is called Teschner's trick [29], we will deduce a non trivial functional relation on the 3 point structure constant of the theory.

Let us finally make some comments about previous studies of LQFT and CFTs. There is a huge physics literature on LQFT for which we refer the reader to the review [?]. On the "axiomatic" side where Ward identities, crossing symmetry of correlation functions and the content of the spectrum are assumed to hold, we refer to [24]. Takhtajan et al. developed (semiclassical) Liouville theory in the so-called background field formalism approach: see [28] for the latest results. In this non probabilistic approach, Liouville field theory is expanded as a formal power series in γ around the minimum of the action (1.3) and the parameter Q in the action is given by its value in classical Liouville theory $Q = \frac{2}{\gamma}$. Ward identities are established in this context.

On the constructive, probabilistic side there are very few results on conformal invariance and Ward identities. For the **Gaussian Free Field** a complete description of the CFT is developed in Kang and Makarov's monograph [20]. In particular, they derive Ward and BPZ identities for this particular CFT with central charge $c_{\text{GFF}} = 1$. Essentially, the free field corresponds to setting the cosmological constant μ to 0. The resulting CFT has a very different structure than the $\mu \neq 0$ case.

For interacting field theories the work of Smirnov and Chelkak-Smirnov [6, 27] provides a nearly complete description of the scaling limit of the **Ising model** at critical temperature and conformal invariance of phase boundary curves and field correlation functions: see Chelkak-Hongler-Izyurov [7] for the latest results. In fact, one can also construct the Ising primary field Φ as a random distribution: see Camia-Garban-Newman [5]. The average of the associated stress-energy tensor (in the case of the Ising model, the product of two neighboring spins) was also rigorously constructed by Hongler and Smirnov [19].

In the context of **SLE and CLE**, which are random conformally invariant curves, one can construct partition functions or correlation functions as probabilities of events related to several SLEs or CLE. The correlation functions correspond to CFTs with central charge $c_{\text{SLE}}, c_{\text{CLE}} \leq 1$. In this context, the understanding of the correlation functions is quite precise though not complete: see Dubédat [11], Bauer-Bernard-Kytola [2] for multiple SLEs and more recently Camia-Gandolfi-Kleban [4] for a construction of primary fields in the context of CLEs. In the context of CLEs, Doyon [10] has proposed a construction of the stress-energy tensor but it relies on assumptions which have not been proved yet.

The rest of the paper is organized as follows. In the next section, we set the notations, recall some results of [8] and state the main results of the paper: the Ward and BPZ identities. In the next section, we give the proofs of these results. In the section L^p estimates, we gather technical estimates we need on the correlation functions for the proofs of the previous section.

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2 Main results

2.1 Background and notations

In this section we recall the precise definition of the Liouville correlation functions (1.2) as given in [8].

Gaussian free field. We represent the Riemann sphere $\hat{\mathbb{C}}$ by stereographic projection as \mathbb{C} and equip it with the round metric. The associated volume measure is then given by $\hat{g}(z)dz$ where dz is the Lebesgue measure and

$$\hat{g}(z) = \frac{4}{(1 + \bar{z}z)^2}.$$

We denote by $X(z)$ the Gaussian Free Field on \mathbb{C} where the additive constant is fixed by $\int_{\mathbb{C}} X(z)\hat{g}(z)dz = 0$ i.e. the Gaussian field with covariance

$$\mathbb{E}X(z)X(z') = G(z, z') = \ln \frac{1}{|z - z'|} - \frac{1}{4}(\ln \hat{g}(z) + \ln \hat{g}(z')) + b \quad (2.1)$$

where $b := \ln 2 - \frac{1}{2}$. We gather some useful formulas on derivatives of the Green function in the appendix.

The field X is distribution valued and we will work with a mollified regularization of the GFF, namely $X_\epsilon = X * \rho_\epsilon$ with $\rho_\epsilon(z) = \frac{1}{\epsilon^2}\rho(\frac{z}{\epsilon})$ where ρ is a smooth mollifier with compact support on $(0, \infty)$. We have

$$\lim_{\epsilon \rightarrow 0} (\mathbb{E}[X_\epsilon(z)^2] + \ln(a\epsilon)) = -\frac{1}{2} \ln \hat{g}(z) \quad (2.2)$$

uniformly on \mathbb{C} where the constant a depends on the regularization. We will use the notation ϕ_ϵ as a shortcut for

$$\phi_\epsilon(z) = ((X + Q/2 \ln \hat{g}) * \rho_\epsilon)(z). \quad (2.3)$$

Gaussian multiplicative chaos. Define the measure

$$M_{\gamma, \epsilon}(dz) := (A\epsilon)^{\frac{\gamma^2}{2}} e^{\gamma \phi_\epsilon(z)} dz \quad (2.4)$$

where the constant³ $A = ae^b$ and ϕ_ϵ is defined in (2.3). For $\gamma \in [0, 2)$, we have the convergence in probability

$$M_\gamma = \lim_{\epsilon \rightarrow 0} M_{\gamma, \epsilon} = e^{\frac{\gamma^2}{2}b} \lim_{\epsilon \rightarrow 0} e^{\gamma X_\epsilon - \frac{\gamma^2}{2} \mathbb{E}[X_\epsilon^2]} \hat{g}(z) dz \quad (2.5)$$

and convergence is in the sense of weak convergence of measures. This limiting measure is non trivial and is a (up to a multiplicative constant) Gaussian multiplicative chaos (see [26] for latest updates) of the field X with respect to the measure $\hat{g}(z)dz$.

Regularized vertex operators are defined as

$$V_{\alpha, \epsilon}(z) = (A\epsilon)^{\frac{\alpha^2}{2}} e^{\alpha(\phi_\epsilon(z) + c)}. \quad (2.6)$$

where $c \in \mathbb{R}$ is a variable representing the "constant mode" of the GFF.

Liouville correlation functions. Let us denote

$$U_N = \{\mathbf{z} = (z_1, \dots, z_N) \in \mathbb{C}^N, z_i \neq z_j \forall i \neq j\}. \quad (2.7)$$

Fix $\mathbf{z} \in U_N$ and weights $\alpha_1, \dots, \alpha_N$ which satisfy the Seiberg bounds (1.1). Let F be a bounded measurable function of the GFF X and the constant mode c . Define the regularized Liouville correlation function by

$$\langle \prod_k V_{\alpha_k, \epsilon}(z_k) F \rangle_\epsilon := \int_{\mathbb{R}} e^{-2Qc} \mathbb{E}[F \prod_k V_{\alpha_k, \epsilon}(z_k) e^{-\mu e^{\gamma c} M_{\gamma, \epsilon}(\mathbb{C})}] dc. \quad (2.8)$$

³This normalization is chosen to match with the standard physics literature.

In [8] it was then shown that the limit

$$\langle \prod_k V_{\alpha_k}(z_k) F \rangle := \lim_{\epsilon \rightarrow 0} \langle \prod_k V_{\alpha_k, \epsilon}(z_k) F \rangle_\epsilon \quad (2.9)$$

exists and is non trivial. (2.9) gives precise meaning to (1.2) and in particular satisfies the conformal invariance property (1.4).

2.2 Ward identities

We will define the stress energy tensor via a regularized version of (1.6). Let

$$T_\epsilon(z) := Q \partial_z^2 \phi_\epsilon(z) - ((\partial_z \phi_\epsilon(z))^2 - \mathbb{E} X_\epsilon(z)^2) \quad (2.10)$$

where X_ϵ refers to the regularization with smooth mollifier. Note that X_ϵ is smooth (a.s.).

Here is the main theorem on the Ward identities:

Theorem 2.1. (a) *The correlation functions $(z_1, \dots, z_N) \mapsto \langle \prod_l V_{\alpha_l}(z_l) \rangle$ are C^1 in the set U_N .*

(b) *The limit*

$$\lim_{\epsilon \rightarrow 0} \langle T_\epsilon(z) \prod_l V_{\alpha_l}(z_l) \rangle := \langle T(z) \prod_l V_{\alpha_l}(z_l) \rangle$$

exists for all $z \in U_N$ and is given by

$$\langle T(z) \prod_l V_{\alpha_l}(z_l) \rangle = \sum_k \frac{\Delta_{\alpha_k}}{(z - z_k)^2} \langle \prod_l V_{\alpha_l}(z_l) \rangle + \sum_k \frac{1}{z - z_k} \partial_{z_k} \langle \prod_l V_{\alpha_l}(z_l) \rangle.$$

By applying the Möbius invariance relation (1.4) with $\psi_{\epsilon, j}(z) = z + \epsilon z^j$ ($j = 0, 1$) $\psi_{\epsilon, 2}(z) = \frac{z}{1 - \epsilon z}$ and letting ϵ go to 0, we get as a corollary the *global Ward identities*

$$\sum_k \partial_{z_k} \langle \prod_l V_{\alpha_l}(z_l) \rangle = 0, \quad \sum_k (z_k \partial_{z_k} + 2\Delta_{\alpha_k}) \langle \prod_l V_{\alpha_l}(z_l) \rangle = 0, \quad \sum_k (z_k^2 \partial_{z_k} + 4\Delta_{\alpha_k} \operatorname{Re}(z_k)) \langle \prod_l V_{\alpha_l}(z_l) \rangle = 0.$$

2.3 The BPZ equations

The main result of this subsection is the following theorem:

Theorem 2.2. *The following differential equation holds in $\mathbb{C} \setminus \{z_1, \dots, z_N\}$*

$$\frac{4}{\gamma^2} \partial_{zz}^2 \langle V_{-\frac{\gamma}{2}}(z) \prod_l V_{\alpha_l}(z_l) \rangle + \sum_k \frac{\Delta_{\alpha_k}}{(z - z_k)^2} \langle V_{-\frac{\gamma}{2}}(z) \prod_l V_{\alpha_l}(z_l) \rangle + \sum_k \frac{1}{z - z_k} \partial_{z_k} \langle V_{-\frac{\gamma}{2}}(z) \prod_l V_{\alpha_l}(z_l) \rangle = 0.$$

One also has the following dual equation

$$\frac{\gamma^2}{4} \partial_{zz}^2 \langle V_{-\frac{2}{\gamma}}(z) \prod_l V_{\alpha_l}(z_l) \rangle + \sum_k \frac{\Delta_{\alpha_k}}{(z - z_k)^2} \langle V_{-\frac{2}{\gamma}}(z) \prod_l V_{\alpha_l}(z_l) \rangle + \sum_k \frac{1}{z - z_k} \partial_{z_k} \langle V_{-\frac{2}{\gamma}}(z) \prod_l V_{\alpha_l}(z_l) \rangle = 0.$$

The "fields" $V_{-\frac{\gamma}{2}}(z)$ and $V_{-\frac{2}{\gamma}}(z)$ are called degenerate fields in the physics literature (see [24] for instance) and the above PDEs are called the second order BPZ equations. These equations are supposed to appear within the general framework of CFT (not just Liouville) discovered in the seminal paper [3]. Only correlation functions which involve degenerate fields are expected to satisfy simple PDEs. The fields $V_{-\frac{\gamma}{2}}(z)$ and $V_{-\frac{2}{\gamma}}(z)$ are in fact the simplest degenerate fields of the theory as they satisfy second order PDEs whereas the other degenerate fields are expected to satisfy higher order PDEs: we will address their study in a forthcoming work.

The BPZ equations, combined with the conformal invariance (1.4), is a powerful tool to characterize the 4 point correlation functions: we will exploit this more precisely in the next subsection.

2.4 On the 4 point correlation function and a symmetry principle for the 3 point structure constant

In this subsection, we will show how to use the previous BPZ equations to deduce an exact expression for the 4 point correlation $\langle V_{-\frac{\gamma}{2}}(z) \prod_{l=1}^3 V_{\alpha_l}(z_l) \rangle$. We will deduce from this expression a non trivial relation on the 3 point function: this is usually referred to as Teschner's trick in the physics literature since it was discovered by Teschner to give a simple derivation of the celebrated DOZZ formula for the 3 point function.

Let us first use Möbius invariance to simplify the three and four point functions. This fixes the three point function up to a constant

$$\langle \prod_{l=1}^3 V_{\alpha_l}(z_l) \rangle = |z_1 - z_2|^{2\Delta_{12}} |z_2 - z_3|^{2\Delta_{23}} |z_1 - z_3|^{2\Delta_{13}} C(\alpha_1, \alpha_2, \alpha_3) \quad (2.11)$$

where we denoted $\Delta_{12} = \Delta_{\alpha_3} - \Delta_{\alpha_1} - \Delta_{\alpha_2}$ etc. Similarly, the four point function is fixed up to a single function depending on the cross ratio of the points. Specializing to the case we are interested in, we get

$$\langle V_{-\frac{\gamma}{2}}(z) \prod_{l=1}^3 V_{\alpha_l}(z_l) \rangle = |z_3 - z|^{-4\Delta_{-\frac{\gamma}{2}}} |z_2 - z_1|^{2(\Delta_3 - \Delta_2 - \Delta_1 - \Delta_{-\frac{\gamma}{2}})} |z_3 - z_1|^{2(\Delta_2 + \Delta_{-\frac{\gamma}{2}} - \Delta_3 - \Delta_1)} \quad (2.12)$$

$$\times |z_3 - z_2|^{2(\Delta_1 + \Delta_{-\frac{\gamma}{2}} - \Delta_3 - \Delta_2)} G\left(\frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)}\right) \quad (2.13)$$

where $\Delta_{-\frac{\gamma}{2}} = -\frac{\gamma}{4}(Q + \frac{\gamma}{4})$ and $\Delta_l = \frac{\alpha_l}{2}(Q - \frac{\alpha_l}{2})$. We can recover C and $G(z)$ as the following limits

$$C(\alpha_1, \alpha_2, \alpha_3) = \lim_{z_3 \rightarrow \infty} |z_3|^{4\Delta_3} \langle V_{\alpha_1}(0) V_{\alpha_2}(1) V_{\alpha_3}(z_3) \rangle \quad (2.14)$$

$$G(z) = \lim_{z_3 \rightarrow \infty} |z_3|^{4\Delta_3} \langle V_{-\frac{\gamma}{2}}(z) V_{\alpha_1}(0) V_{\alpha_2}(1) V_{\alpha_3}(z_3) \rangle. \quad (2.15)$$

In order to state the result, we introduce

$$F_-(z) = {}_2F_1(a, b, c, z), \quad F_+(z) = z^{1-c} {}_2F_1(1+a-c, 1+b-c, 2-c, z) = z^{\frac{\gamma}{2}(Q-\alpha_1)} {}_2F_1(1+a-c, 1+b-c, 2-c, z)$$

where ${}_2F_1(a, b, c, z)$ are the standard hypergeometric series extended to $\mathbb{C} \setminus]1, \infty[$ and the real parameters a, b, c have the following expression

$$a = \frac{\gamma}{2}(\frac{\alpha_1}{2} - \frac{Q}{2}) + \frac{\gamma}{2}(\frac{\alpha_2}{2} + \frac{\alpha_3}{2} - \frac{\gamma}{2}) - \frac{1}{2} \quad b = \frac{\gamma}{2}(\frac{\alpha_1}{2} - \frac{Q}{2}) + \frac{\gamma}{2}(\frac{\alpha_2}{2} - \frac{\alpha_3}{2}) + \frac{1}{2} \quad (2.16)$$

and

$$c = 1 + \frac{\gamma}{2}(\alpha_1 - Q). \quad (2.17)$$

Theorem 2.3. *Let $(\alpha_1, \alpha_2, \alpha_3)$ be such that we have $Q - \frac{1}{\gamma} < \alpha_1 < Q - \frac{\gamma}{2}$ and $\sum_l \alpha_l > 2Q + \frac{\gamma}{2}$. Then the function G has the following explicit expression*

$$\frac{G(z)}{|z|^{\frac{\gamma\alpha_1}{2}} |z-1|^{\frac{\gamma\alpha_2}{2}}} = C(\alpha_1 - \frac{\gamma}{2}, \alpha_2, \alpha_3) |F_-(z)|^2 - \mu \frac{\pi}{l(-\frac{\gamma^2}{4}) l(\frac{\gamma\alpha_1}{2}) l(2 + \frac{\gamma^2}{4} - \frac{\gamma\alpha_1}{2})} C(\alpha_1 + \frac{\gamma}{2}, \alpha_2, \alpha_3) |F_+(z)|^2$$

where $l(x) = \frac{\Gamma(x)}{\Gamma(1-x)}$.

The conditions on $(\alpha_1, \alpha_2, \alpha_3)$ are just technical and could be relaxed with more refined estimates in the proof; one can also notice that the set of $(\alpha_1, \alpha_2, \alpha_3)$ that satisfy the assumptions of Theorem 2.3 is non empty for $\gamma < \sqrt{2}$.

From this we can deduce the following corollary on the 3 point structure constants:

Corollary 2.4. *Let $(\alpha_1, \alpha_2, \alpha_3)$ be such that for $i = 1$ and $i = 2$ we have $Q - \frac{1}{\gamma} < \alpha_i < Q - \frac{\gamma}{2}$ and $\sum_l \alpha_l > 2Q + \frac{\gamma}{2}$. Then we have the following relation*

$$\frac{C(\alpha_1 + \frac{\gamma}{2}, \alpha_2, \alpha_3)}{C(\alpha_1 - \frac{\gamma}{2}, \alpha_2, \alpha_3)} = -\frac{1}{\pi\mu} \frac{l(-\frac{\gamma^2}{4}) l(\frac{\gamma\alpha_1}{2}) l(\frac{\alpha_1\gamma}{2} - \frac{\gamma^2}{4}) l(\frac{\gamma}{4}(-\alpha_1 + \alpha_2 + \alpha_3 - \frac{\gamma}{2}))}{l(\frac{\gamma}{4}(\alpha_1 + \alpha_2 + \alpha_3 - \frac{\gamma}{2} - 2Q)) l(\frac{\gamma}{4}(\alpha_1 + \alpha_2 - \alpha_3 - \frac{\gamma}{2})) l(\frac{\gamma}{4}(\alpha_1 + \alpha_3 - \alpha_2 - \frac{\gamma}{2}))}.$$

3 Proofs

3.1 Integrability properties of Liouville correlations

In Section 4 we prove detailed estimates for the Liouville correlations $\langle \prod_k V_{\alpha_k}(z_k) \rangle$ as some of the points get together. For the proof of the Ward identities we need the following special cases. Let the weights $\alpha_1, \dots, \alpha_N$ satisfy the Seiberg bounds and $\mathbf{z} = (z_1, \dots, z_N) \in U_N$. Define for $x, y \in \mathbb{C}$

$$f_{\epsilon, \mathbf{z}}(x) := \langle V_{\gamma, \epsilon}(x) \prod_k V_{\alpha_k, \epsilon}(z_k) \rangle_{\epsilon}, \quad f_{\mathbf{z}}(x) := \sup_{\epsilon > 0} f_{\epsilon, \mathbf{z}}(x) \quad (3.1)$$

$$F_{\epsilon, \mathbf{z}}(x, y) := \langle V_{\gamma, \epsilon}(x) V_{\gamma, \epsilon}(y) \prod_k V_{\alpha_k, \epsilon}(z_k) \rangle_{\epsilon}, \quad F_{\mathbf{z}}(x, y) := \sup_{\epsilon > 0} F_{\epsilon, \mathbf{z}}(x, y). \quad (3.2)$$

and denote

$$O_{\delta}(\mathbf{z}) := \mathbb{C} \setminus \cup_i B(z_i, \delta). \quad (3.3)$$

Then:

Proposition 3.1. a) Let $\epsilon > 0$. Then $f_{\epsilon, \mathbf{z}}$ and $F_{\epsilon, \mathbf{z}}$ are smooth. Moreover $\|f_{\epsilon, \mathbf{z}}\|_{\infty} < \infty$ and $|F_{\epsilon, \mathbf{z}}(x, y)| \leq C_{\epsilon}(1 + |x|)^{-p}(1 + |y|)^{-p}$ where $p > 2$.
 (b) $f_{\mathbf{z}}$ and $F_{\mathbf{z}}$ belong to $L^p(\mathbb{C})$ and $L^p(\mathbb{C}^2)$ respectively for some $p > 1$ and

$$\int_{\mathbb{C}} (1 + |\ln |y - z_i||)^k f_{\mathbf{z}}(x) dx < \infty \quad i = 1, \dots, N \quad (3.4)$$

$$\int_{\mathbb{C}^2} (1 + |\ln |x - y||)^k F_{\mathbf{z}}(x, y) dxdy < \infty \quad (3.5)$$

for all k . These results hold uniformly in \mathbf{z} on compact subsets of U_N .

(c) Let $x, y \in O_{\delta}(\mathbf{z})$. Then

$$F_{\mathbf{z}}(x, y) \leq C_{\delta} |x - y|^{-2+\zeta}$$

for some $\zeta > 0$.

3.2 Integration by parts and a KPZ identity

Let $f \in C_0^{\infty}(\mathbb{C})$ and set $X(f) = \int X(z)f(z)dz$. The following identity follows by integration by parts in the Gaussian measure:

$$\begin{aligned} \langle X(f) \prod_k V_{\alpha_k, \epsilon}(z_k) \rangle_{\epsilon} &= \sum_i \alpha_i \mathbb{E}(X(f) X_{\epsilon}(z_i)) \langle \prod_k V_{\alpha_k, \epsilon}(z_k) \rangle_{\epsilon} \\ &\quad - \mu \gamma \int_{\mathbb{C}} \mathbb{E}(X(f) X_{\epsilon}(y)) \langle V_{\gamma, \epsilon}(y) \prod_k V_{\alpha_k, \epsilon}(z_k) \rangle_{\epsilon} dy. \end{aligned} \quad (3.6)$$

By Proposition 3.1 the y integral converges since $|\mathbb{E}(X(f) X_{\epsilon}(y))| \leq C \ln(2 + |y|)$.

As an application we compute the derivative of the regularized correlations. Let

$$r_{\epsilon} = \rho_{\epsilon} * \ln \hat{g}.$$

Then

$$\partial_{z_i} \langle \prod_k V_{\alpha_k, \epsilon}(z_k) \rangle_{\epsilon} = \alpha_i \langle (\partial_z X_{\epsilon}(z_i) + \frac{\gamma}{2} \partial_z r_{\epsilon}(z_i)) \prod_k V_{\alpha_k, \epsilon}(z_k) \rangle_{\epsilon}.$$

Let us denote

$$\frac{1}{(z)_{\epsilon}} := \int_{\mathbb{C}^2} \frac{1}{z + u - v} \rho_{\epsilon}(u) \rho_{\epsilon}(v) dudv.$$

Then

$$\mathbb{E}(\partial_z X_\epsilon(z) X_\epsilon(z')) = -\frac{1}{2} \frac{1}{(z - z')_\epsilon} - \frac{1}{4} \partial_z r_\epsilon(z)$$

and using (3.6) we get

$$\begin{aligned} \partial_{z_i} \langle \prod_k V_{\alpha_k, \epsilon}(z_k) \rangle_\epsilon &= \frac{\alpha_i}{4} (2Q - \sum_j \alpha_j) \partial_z r_\epsilon(z_i) \langle \prod_k V_{\alpha_k, \epsilon}(z_k) \rangle_\epsilon - \frac{1}{2} \alpha_i \sum_{j \neq i} \alpha_j \frac{1}{(z_i - z_j)_\epsilon} \langle \prod_k V_{\alpha_k, \epsilon}(z_k) \rangle_\epsilon \\ &\quad + \frac{1}{2} \alpha_i \mu \gamma \int_{\mathbb{C}} \left(\frac{1}{(z_i - y)_\epsilon} + \frac{1}{2} \partial_z r_\epsilon(z_i) \right) \langle V_{\gamma, \epsilon}(y) \prod_k V_{\alpha_k, \epsilon}(z_k) \rangle_\epsilon dy \end{aligned} \quad (3.7)$$

where we used $\mathbb{E}(\partial_z X_\epsilon(z) X_\epsilon(z)) = \frac{1}{2} \partial_z \mathbb{E}(X_\epsilon(z)^2) = -\frac{1}{4} \partial_z r_\epsilon$. (3.7) simplifies due to the following identity

Lemma 3.2. *For all $\epsilon \geq 0$*

$$\gamma \mu \int \langle V_{\gamma, \epsilon}(y) \prod_k V_{\alpha_k, \epsilon}(z_k) \rangle_\epsilon dy = \left(\sum_i \alpha_i - 2Q \right) \langle \prod_k V_{\alpha_k, \epsilon}(z_k) \rangle_\epsilon. \quad (3.8)$$

Proof. Recalling the c -dependence in (2.6) we get by a simple change of variables $\gamma^{-1} \ln \mu + c = c'$ that

$$\begin{aligned} \langle \prod_k V_{\alpha_k, \epsilon}(z_k) \rangle_{\hat{g}, \epsilon, \mu} &= \int_{\mathbb{R}} e^{-2Qc} \mathbb{E} \left[\prod_k V_{\alpha_k, \epsilon}(z_k) e^{-\mu \int_{\mathbb{C}} V_{\gamma, \epsilon}(x) dx} \right] dc \\ &= \mu^{-\frac{\sum_i \alpha_i - 2Q}{\gamma}} \int_{\mathbb{R}} e^{-2Qc'} \mathbb{E} \left[\prod_k V_{\alpha_k, \epsilon}(z_k) e^{-\int_{\mathbb{C}} V_{\gamma, \epsilon}(x) dx} \right] dc'. \end{aligned}$$

The identity follows by differentiating in μ . The limit as $\epsilon \rightarrow 0$ follows in virtue of Proposition 3.1. \square

Combining the Lemma with (3.7) we end up with

$$\partial_{z_i} \langle \prod_k V_{\alpha_k, \epsilon}(z_k) \rangle_\epsilon = -\frac{1}{2} \sum_{j \neq i} \frac{\alpha_i \alpha_j}{(z_i - z_j)_\epsilon} \langle \prod_k V_{\alpha_k, \epsilon}(z_k) \rangle_\epsilon + \frac{1}{2} \alpha_i \mu \gamma Y_{i, \epsilon}(\mathbf{z}) \quad (3.9)$$

with

$$Y_{i, \epsilon}(\mathbf{z}) = \int_{\mathbb{C}} \frac{1}{(z_i - y)_\epsilon} \langle V_{\gamma, \epsilon}(y) \prod_k V_{\alpha_k, \epsilon}(z_k) \rangle_\epsilon dy. \quad (3.10)$$

Note that Proposition 3.1 does not allow us to control the limit as ϵ tends to zero of this expression since it guarantees only integrability of a logarithmic singularity in $z_i - y$.

3.3 Proof of Theorem 2.1 (a)

We will prove convergence of $Y_{i, \epsilon}(\mathbf{z})$ as $\epsilon \rightarrow 0$ by expressing it in terms of integrals of other correlation functions that can be controlled using Proposition 3.1.

Let us fix the point $\mathbf{z} \in U_N$ and suppress it in the notation i.e. we denote $f_{\epsilon, \mathbf{z}}(x)$ by $f_\epsilon(x)$ etc. Recall that f_ϵ is smooth and by Proposition 3.1 integrable. Hence its Beurling transform (see Appendix)

$$A_\epsilon(z) = (\mathcal{B}f_\epsilon)(z) = -\lim_{\eta \rightarrow 0} \frac{1}{\pi} \int_{\mathbb{C}} \frac{1}{(z - y)^2} f_\epsilon(y) 1_{|z - y| \geq \eta} dy$$

is continuous and satisfies in the distributional sense

$$\partial_{\bar{z}} A_\epsilon(z) = \partial_z f_\epsilon(z) = \partial_z \langle V_{\gamma, \epsilon}(z) \prod_k V_{\alpha_k, \epsilon}(z_k) \rangle_\epsilon.$$

Proposition 3.1 $f_\epsilon \in L^p(\mathbb{C})$ hence its Beurling transform $A_\epsilon \in L^p(\mathbb{C})$ (see Appendix). Using the integration by parts identity (3.9) and recalling the definition (3.1) we then get

$$\partial_{\bar{z}} A^\epsilon(z) = -\frac{1}{2} \gamma \sum_i \frac{\alpha_i}{(z - z_i)_\epsilon} f_\epsilon(z) + \frac{1}{2} \mu \gamma^2 \int_{\mathbb{C}} \frac{1}{(z - y)_\epsilon} F_\epsilon(z, y) dy. \quad (3.11)$$

The function $z \rightarrow \frac{1}{(z-z_i)_\epsilon} f_\epsilon(z)$ is smooth and integrable by Proposition 3.1. Hence its Cauchy transform

$$b_{i,\epsilon}(z) =: \frac{1}{\pi} \int_{\mathbb{C}} \frac{1}{(z-y)} \frac{1}{(y-z_i)_\epsilon} f_\epsilon(y) dy$$

is continuous and satisfies in the distributional sense

$$\partial_{\bar{z}} b_{i,\epsilon}(z) = \frac{1}{(z-z_i)_\epsilon} f_\epsilon(z).$$

Furthermore, splitting the integral to $|y-z| < 1$ and $|y-z| \geq 1$ we get

$$|b_{i,\epsilon}(z)| \leq C_{\epsilon,z_i} \left(\|f_\epsilon\|_\infty (1+|z|)^{-1} + \int_{\mathbb{C}} (1+|z-y|)^{-1} f_\epsilon(y) dy \right).$$

Hence, by $\|f_\epsilon\|_1 < \infty$ and dominated convergence theorem

$$b_{i,\epsilon}(z) \rightarrow 0 \quad \text{as } |z| \rightarrow \infty.$$

Summarizing:

$$\frac{1}{2} \gamma \sum_i \frac{\alpha_i}{(z-z_i)_\epsilon} f_\epsilon(z) = \frac{1}{2} \gamma \sum_i \alpha_i \partial_{\bar{z}} b_{i,\epsilon}(z) =: \partial_{\bar{z}} B_\epsilon(z)$$

where $B_\epsilon(z)$ is a continuous function vanishing at infinity.

Consider finally the second term on the RHS of (3.11). It is continuous since the two other terms are. Its Cauchy transform

$$\frac{\mu\gamma^2}{2\pi} \int_{\mathbb{C}} \frac{1}{(z-y)} \int_{\mathbb{C}} \frac{1}{(y-x)_\epsilon} F_\epsilon(y,x) dx dy =: -C_\epsilon(z)$$

is bounded by Proposition 3.1:

$$|C_\epsilon(z)| \leq C_\epsilon \int \frac{1}{|z-y|} \frac{1}{1+|x-y|} \frac{1}{1+|x|^p} \frac{1}{1+|y|^p} dx dy.$$

It is easy to show then that $C_\epsilon(z) \rightarrow 0$ as $|z| \rightarrow \infty$. Thus the second term on the RHS of (3.11) equals $-\partial_{\bar{z}} C_\epsilon(z)$ and (3.11) becomes the identity

$$\partial_{\bar{z}}(A_\epsilon(z) + B_\epsilon(z) + C_\epsilon(z)) = 0$$

i.e. $A_\epsilon + B_\epsilon + C_\epsilon$ is an analytic function on \mathbb{C} . Since B_ϵ and C_ϵ vanish at infinity and A_ϵ is in $L^p(\mathbb{C})$ we obtain

$$A_\epsilon + B_\epsilon + C_\epsilon = 0. \tag{3.12}$$

To connect this identity to $Y_{i,\epsilon}$ defined in (3.10) we use $\frac{1}{z-y} = \frac{1}{z-z_i} \left(\frac{y-z_i}{z-y} + 1 \right)$ to write

$$b_{i,\epsilon}(z) = \frac{1}{z-z_i} (D_{i,\epsilon}(z) - \frac{1}{\pi} Y_{i,\epsilon}) \tag{3.13}$$

where

$$D_{i,\epsilon}(z) = \frac{1}{\pi} \int_{\mathbb{C}} \frac{z_i - y}{z - y} \frac{1}{(z_i - y)_\epsilon} f_\epsilon(y) dy.$$

Hence we arrive at

$$\frac{\gamma}{2\pi} \sum_i \frac{Y_{i,\epsilon}}{z-z_i} = A_\epsilon(z) + C_\epsilon(z) + \frac{1}{2} \gamma \sum_i \frac{D_{i,\epsilon}(z)}{z-z_i} \equiv E_\epsilon(z). \tag{3.14}$$

Convergence of $Y_{i,\epsilon}$ follows if we prove convergence of E_ϵ in $L^1_{loc}(\mathbb{C})$ since we may write

$$\gamma Y_{i,\epsilon} = \frac{1}{r} \int_{B(0,r)} E_\epsilon(z_i + z) \frac{z}{|z|} dz$$

for r small enough. To prove convergence of E_ϵ in $L^1_{loc}(\mathbb{C})$ consider the three terms in turn.

By Proposition 3.1 f_ϵ converges in L^p for some $p > 1$. Hence its Beurling transform A_ϵ converges in L^p and thus in $L^1_{loc}(\mathbb{C})$.

Next, consider $C_\epsilon(z)$. Symmetrizing we have

$$C_\epsilon(z) = -\frac{\mu\gamma^2}{4\pi} \int_{\mathbb{C}^2} \frac{y-x}{(z-y)(z-x)} \frac{1}{(y-x)_\epsilon} F_\epsilon(x,y) dx dy.$$

The integrand is bounded by $\frac{\sup_\epsilon |F_\epsilon(x,y)|}{|z-y||z-x|}$ and for compact K

$$\int_K \left(\int_{\mathbb{C}^2} \frac{\sup_\epsilon |F_\epsilon(x,y)|}{|z-y||z-x|} dx dy \right) dz \leq C_K \int \int (|\ln|y-x|| + 1) \sup_\epsilon |F_\epsilon(x,y)| dx dy \leq C'_K \quad (3.15)$$

by Proposition 3.1. Hence by dominated convergence theorem C_ϵ converges in $L^1_{loc}(\mathbb{C})$.

Finally

$$\left| \frac{D_{i,\epsilon}(z)}{z-z_i} \right| \leq C \frac{1}{|z-z_i|} \int_{\mathbb{C}} \frac{1}{|z-y|} \sup_\epsilon |f_\epsilon(y)| dy$$

and by Proposition 3.1

$$\int_K \frac{1}{|z-z_i|} \int_{\mathbb{C}} \frac{1}{|z-y|} \sup_\epsilon |f_\epsilon(y)| dy dz \leq C_K \int_{\mathbb{C}} (|\ln|y-z_i|| + 1) \sup_\epsilon |f_\epsilon(y)| dy \leq C'_K.$$

Hence by dominated convergence theorem $\frac{D_{i,\epsilon}(z)}{z-z_i}$ converges in $L^1_{loc}(\mathbb{C})$. We have thus proved the pointwise convergence of the continuous functions $Y_{i,\epsilon}(z_1, \dots, z_N)$ for $(z_1, \dots, z_N) \in U_N$ towards a limit denoted $Y_i(z_1, \dots, z_N)$. The convergence is uniform on compact subsets of U_N since the claims in Proposition 3.1 (b) are. Consequently B_ϵ converges also in $L^1_{loc}(\mathbb{C})$ towards a limit denoted by B as $\epsilon \rightarrow 0$. Combining with (3.9) the proof of Theorem 2.1 (a) is finished. \square

As a corollary to Theorem 2.1 (a) we note that $f(z)$ is C^1 in $O_\delta(\mathbf{z})$ and so its Beurling transform is continuous there. Hence the function $A = \lim_{\epsilon \rightarrow 0} A_\epsilon$ defined in L^1_{loc} is actually continuous in $O_\delta(\mathbf{z})$. Similarly, D_i is the Cauchy transform of f and hence continuous. Thus B is continuous in $O_\delta(\mathbf{z})$. Therefore $C = -A - B$ also extends to a continuous function in $O_\delta(\mathbf{z})$. It is given by

$$C(z) = -\frac{\mu\gamma^2}{4\pi} \int_{\mathbb{C}^2} \frac{1}{(z-y)(z-x)} F(x,y) dx dy.$$

Finally, the equation (3.9) holds at $\epsilon = 0$ and leads to

$$\begin{aligned} \sum_i \frac{1}{z-z_i} \partial_{z_i} \langle \prod_k V_{\alpha_k}(z_k) \rangle &= -\frac{1}{2} \sum_{i \neq j} \frac{\alpha_i \alpha_j}{(z-z_i)(z-z_j)} \langle \prod_k V_{\alpha_k}(z_k) \rangle + \frac{1}{2} \mu \gamma \sum_i \frac{\alpha_i Y_i}{z-z_i} \\ &= -\frac{1}{4} \sum_{i \neq j} \frac{\alpha_i \alpha_j}{(z-z_i)(z-z_j)} \langle \prod_k V_{\alpha_k}(z_k) \rangle + \frac{1}{2} \mu \gamma \sum_i \frac{\alpha_i}{z-z_i} \int \frac{1}{z-y} f(y) dy - \mu \pi B(z). \end{aligned} \quad (3.16)$$

3.4 Proof of Theorem 2.1 (b)

We start with the second term in (2.10). Proceeding as in in Section 3.2 with the $\partial_z \phi_\epsilon$ insertion we get

$$\langle (\partial_{zz}^2 \phi_\epsilon(z)) \prod_l V_{\alpha_l}(z_l) \rangle = \sum_i \alpha_i \partial_{zz}^2 D_\epsilon(z-z_i) \langle \prod_l V_{\alpha_l}(z_l) \rangle - \mu \gamma \int_{\mathbb{C}} \partial_{zz}^2 D_\epsilon(z-y) f(y) dy. \quad (3.17)$$

where the Lemma 3.2 was used (at $\epsilon = 0$) again to cancel the terms proportional to $\partial_z^2 r$ and we denoted

$$D_\epsilon(z) = \int \rho_\epsilon(z-y) \ln |y|^{-1} dy.$$

Next consider the first term in (2.10). Integrating by parts twice gives

$$\begin{aligned} & \langle (\partial_z \phi_\epsilon(z))^2 - \mathbb{E}[(\partial_z X_\epsilon(z))^2] \prod_l V_{\alpha_l}(z_l) \rangle \\ &= \sum_{k=1} \alpha_k \partial_z D_\epsilon(z - z_k) \langle \partial_z \phi_\epsilon(z) \prod_l V_{\alpha_l}(z_l) \rangle - \mu \gamma \int_{\mathbb{C}} \partial_z D_\epsilon(z-y) \langle \partial_z \phi_\epsilon(z) V_\gamma(y) \prod_l V_{\alpha_l}(z_l) \rangle dy \\ &= \sum_{j,k} \alpha_j \alpha_k \partial_z D_\epsilon(z - z_k) \partial_z D_\epsilon(z - z_j) \langle \prod_l V_{\alpha_l}(z_l) \rangle \\ &\quad - 2\mu \gamma \sum_k \alpha_k \partial_z D_\epsilon(z - z_k) \int_{\mathbb{C}} \partial_z D_\epsilon(z-y) f(y) dy - \mu \gamma^2 \int_{\mathbb{C}} (\partial_z D_\epsilon(z-y))^2 f(y) dy \\ &\quad + \mu^2 \gamma^2 \int_{\mathbb{C}} \partial_z D_\epsilon(z-y) \partial_z D_\epsilon(z-x) F(x,y) dy dx. \end{aligned} \tag{3.18}$$

Recalling that $\rho_\epsilon(z) = \frac{1}{\epsilon^2} \rho(\frac{z}{\epsilon^2})$ where ρ is a smooth mollifier with compact support on $(0, \infty)$, we compute

$$\begin{aligned} \partial_z D_\epsilon(z) &= -\frac{1}{2z} (1 - \chi_1(|z|^2/\epsilon^2)) \\ \partial_{zz}^2 D_\epsilon(z) &= \frac{1}{2z^2} (1 - \chi_2(|z|^2/\epsilon^2)) \end{aligned}$$

with $\chi_i \in C_0^\infty([0, \infty))$, $\chi_i(0) = 1$, (explicitely $\chi_1 = \int \rho 1_{|y| > |z|}$ and $\chi_2 = \chi_1 + \pi |z|^2 \rho$).

Consider the limits as $\epsilon \rightarrow 0$ of the various terms in (3.17) and (3.18). For instance

$$\int_{\mathbb{C}} (\partial_{zz}^2 D_\epsilon(z-y) - \frac{1}{2(z-y)^2} 1_{|z-y| \geq \epsilon}) f(y) dy = \frac{1}{2} \int e^{-2i\theta} f(z + \epsilon r e^{i\theta}) (1_{r < 1} - \chi_2(r^2)) \frac{dr}{r} d\theta$$

tends to zero as $\epsilon \rightarrow 0$ since $f(y)$ is C^1 on $O_\delta(\mathbf{z})$. Thus

$$\lim_{\epsilon \rightarrow 0} \int_{\mathbb{C}} \partial_{zz}^2 D_\epsilon(z-y) f(y) dy = -\frac{\pi}{2} (\mathcal{B}f)(z).$$

In the same way

$$\lim_{\epsilon \rightarrow 0} \int_{\mathbb{C}} (\partial_z D_\epsilon(z-y))^2 f(y) dy = -\frac{\pi}{4} (\mathcal{B}f)(z).$$

For the last term in (3.18) we get

$$\begin{aligned} & \left| \int_{\mathbb{C}} \left(\frac{1}{4} \frac{1}{z-y} \frac{1}{z-x} - \partial_z D_\epsilon(z-y) \partial_z D_\epsilon(z-x) \right) F(x,y) dy dx \right| \\ &= \frac{1}{4} \epsilon^2 \int_{\mathbb{C}} \frac{1}{|uv|} (2\chi_1(u) + \chi_1(u)\chi_1(v)) F(z + \epsilon u, z + \epsilon v) du dv \\ &\leq C \epsilon^\zeta \int \chi_1(u) \frac{1}{|v|} |u-v|^{-2+\zeta} du dv + C \epsilon^2 \|F\|_1 \leq C \epsilon^\zeta \end{aligned}$$

where we used Proposition 3.1. Therefore

$$-\mu \gamma^2 \lim_{\epsilon \rightarrow 0} \int_{\mathbb{C}} \partial_z D_\epsilon(z-y) \partial_z D_\epsilon(z-x) F(x,y) dy dx = C(z).$$

Taking $\epsilon \rightarrow 0$ limits of equations (3.17) and (3.18) we end up with

$$\begin{aligned} \langle T(z) \prod_l V_{\alpha_l}(z_l) \rangle &= -\frac{1}{4} \sum_{i,j} \alpha_i \alpha_j \frac{1}{z-z_i} \frac{1}{z-z_j} \langle \prod_l V_{\alpha_l}(z_l) \rangle \\ &\quad + \frac{1}{2} \mu \gamma \sum_k \frac{\alpha_k}{z-z_k} \int_{\mathbb{C}} \frac{1}{z-y} f(y) dy + \mu \pi (\mathcal{B}f)(z) \\ &\quad + \frac{1}{2} Q \sum_i \frac{\alpha_i}{(z-z_i)^2} + \mu \pi C(z) \end{aligned}$$

Finally, combining this with eq. (3.9) and using $B + C = -A = -\mathcal{B}f$ the Ward identity follows. \square

3.5 Holomorphic identity for BPZ

To prepare for the proof of Theorem 2.2 we will consider the identity (3.12) when one of the insertion points has weight $-\frac{\gamma}{2}$ and is evaluated at z . Thus, if we write the insertion points and weights e.g. in A_ϵ explicitly as $A_\epsilon(z, (\alpha_1, z_1), \dots, (\alpha_n, z_n))$ we wish to study $A_\epsilon(z, (-\frac{\gamma}{2}, z), (\alpha_1, z_1), \dots, (\alpha_n, z_n))$. We define

$$\begin{aligned} \bar{A}_\epsilon(z) &= -\frac{1}{\pi} \int \frac{1}{(z-y)^2} \left(\langle V_{-\frac{\gamma}{2}, \epsilon}(z) V_{\gamma, \epsilon}(y) \prod_k V_{\alpha_k, \epsilon}(z_k) \rangle_\epsilon - 1_{|y-z| \leq 1} \langle V_{-\frac{\gamma}{2}, \epsilon}(z) V_{\gamma, \epsilon}(z) \prod_k V_{\alpha_k, \epsilon}(z_k) \rangle_\epsilon \right) dy \\ &\quad + \frac{\gamma^2}{4\pi} \int \frac{1}{(z-y)} \frac{1}{(z-y)_\epsilon} \langle V_{-\frac{\gamma}{2}, \epsilon}(z) V_{\gamma, \epsilon}(y) \prod_k V_{\alpha_k, \epsilon}(z_k) \rangle_\epsilon dy, \end{aligned} \quad (3.19)$$

$$\bar{B}_\epsilon(z) = \frac{\gamma}{2\pi} \sum_i \alpha_i \int \frac{1}{(z-y)} \frac{1}{(y-z_i)_\epsilon} \langle V_{-\frac{\gamma}{2}, \epsilon}(z) V_{\gamma, \epsilon}(y) \prod_k V_{\alpha_k, \epsilon}(z_k) \rangle_\epsilon dy, \quad (3.20)$$

$$\bar{C}_\epsilon(z) = -\frac{\mu\gamma^2}{4\pi} \int \frac{y-x}{(z-y)(z-x)} \frac{1}{(y-x)_\epsilon} \langle V_{-\frac{\gamma}{2}, \epsilon}(z) V_{\gamma, \epsilon}(y) V_{\gamma, \epsilon}(x) \prod_k V_{\alpha_k, \epsilon}(z_k) \rangle_\epsilon dx dy. \quad (3.21)$$

In (3.19) we have written the Beurling transform for the smooth function directly without a limiting procedure and we have also moved the z dependent part of B^ϵ of the previous section into $\bar{A}^\epsilon(z)$. We get then the identity

$$\bar{A}_\epsilon(z) + \bar{B}_\epsilon(z) + \bar{C}_\epsilon(z) = 0.$$

The $\epsilon \rightarrow 0$ limits are covered by the following Lemmas proven in Section 4.4:

Lemma 3.3. *The function $y \mapsto \frac{1}{(z-y)^2} \langle V_{-\frac{\gamma}{2}}(z) V_\gamma(y) \prod_k V_{\alpha_k}(z_k) \rangle$ is integrable for all $z \in \mathbb{C} \setminus \{z_1, \dots, z_n\}$. Define*

$$\bar{A}(z) = -(1 - \frac{\gamma^2}{4}) \frac{1}{\pi} \int \frac{1}{(z-y)^2} \langle V_{-\frac{\gamma}{2}}(z) V_\gamma(y) \prod_k V_{\alpha_k}(z_k) \rangle dy.$$

Then $\lim_{\epsilon \rightarrow 0} \bar{A}_\epsilon = \bar{A}$ where the limit is in $\mathcal{D}'(\mathbb{C} \setminus \{z_1, \dots, z_n\})$.

Lemma 3.4. *The function $(x, y) \mapsto \frac{1}{(z-y)(z-x)} \langle V_{-\frac{\gamma}{2}}(z) V_\gamma(y) V_\gamma(x) \prod_k V_{\alpha_k}(z_k) \rangle$ is integrable for all $z \in \mathbb{C} \setminus \{z_1, \dots, z_n\}$. Define*

$$\bar{C}(z) = -\frac{\mu\gamma^2}{4\pi} \int \frac{1}{(z-x)(z-y)} \langle V_{-\frac{\gamma}{2}}(z) V_\gamma(y) V_\gamma(x) \prod_k V_{\alpha_k}(z_k) \rangle dx dy$$

Then $\lim_{\epsilon \rightarrow 0} \bar{C}_\epsilon = \bar{C}$ where the limit is in $\mathcal{D}'(\mathbb{C} \setminus \{z_1, \dots, z_n\})$.

The main lemma is now the following:

Lemma 3.5. *The following identity holds for all $z \in \mathbb{C} \setminus \{z_1, \dots, z_n\}$*

$$\bar{A}(z) + \bar{B}(z) + \bar{C}(z) = 0$$

where \bar{A}, \bar{C} have been defined in the previous two lemmas and \bar{B} is given by the following expression

$$\bar{B}(z) = \frac{\gamma}{2\pi} \sum_i \frac{\alpha_i}{(z - z_i)} \left(\int \frac{1}{z - y} \langle V_{-\frac{\gamma}{2}}(z) V_\gamma(y) \prod_k V_{\alpha_k}(z_k) \rangle_{\hat{g}} dy - Y_i(z, z_1, \dots, z_n) \right) \quad (3.22)$$

where the functions Y_i are defined by the following formula

$$\partial_{z_i} \langle \prod_l V_{-\frac{\gamma}{2}}(z) V_{\alpha_l}(z_l) \rangle = \alpha_i \frac{\gamma}{4} \frac{1}{z_i - z} - \frac{1}{2} \alpha_i \sum_{k \neq i} \frac{\alpha_k}{z_i - z_k} \langle \prod_l V_{\alpha_l}(z_l) \rangle + \frac{\mu \gamma \alpha_i}{2} Y_i(z, z_1, \dots, z_N)$$

3.6 Proof of Theorem 2.2

From eq. (3.9) we get at $\epsilon = 0$ and $z \in O_\delta(\mathbf{z})$

$$\partial_z \langle V_{-\frac{\gamma}{2}}(z) \prod_l V_{\alpha_l}(z_l) \rangle = \mathcal{F}_1(z) - \frac{\mu \gamma^2}{4} \mathcal{F}_2(z), \quad (3.23)$$

$$\text{with } \mathcal{F}_1(z) := \frac{\gamma}{4} \sum_i \frac{\alpha_i}{(z - z_i)} \langle V_{-\frac{\gamma}{2}}(z) \prod_l V_{\alpha_l}(z_l) \rangle, \quad (3.24)$$

$$\mathcal{F}_2(z) := \int \frac{1}{z - y} \langle V_{-\frac{\gamma}{2}}(z) V_\gamma(y) \prod_l V_{\alpha_l}(z_l) \rangle dy. \quad (3.25)$$

\mathcal{F}_1 is in $C^1(O_\delta(\mathbf{z}))$ and using (3.9) its derivative is given by

$$\begin{aligned} \partial_z \mathcal{F}_1(z) &= -\frac{\gamma}{4} \sum_i \frac{\alpha_i}{(z - z_i)^2} \langle V_{-\frac{\gamma}{2}}(z) \prod_l V_{\alpha_l}(z_l) \rangle, \\ &+ \frac{\gamma^2}{16} \sum_{i,j} \frac{\alpha_i \alpha_j}{(z - z_i)(z - z_j)} \langle V_{-\frac{\gamma}{2}}(z) \prod_l V_{\alpha_l}(z_l) \rangle \\ &- \frac{\mu \gamma^3}{16} \sum_i \frac{\alpha_i}{(z - z_i)} \int \frac{1}{z - y} \langle V_{-\frac{\gamma}{2}}(z) V_\gamma(y) \prod_l V_{\alpha_l}(z_l) \rangle dy. \end{aligned}$$

We only know \mathcal{F}_2 to be in $C(O_\delta(\mathbf{z}))$. We will now compute its derivative in the sense of distributions. Recall that $\mathcal{F}_2 = \lim_{\epsilon \rightarrow 0} \mathcal{F}_{2,\epsilon}$ where

$$\mathcal{F}_{2,\epsilon}(z) = \int \frac{1}{(z - y)_\epsilon} \langle V_{-\frac{\gamma}{2},\epsilon}(z) V_{\gamma,\epsilon}(y) \prod_l V_{\alpha_l,\epsilon}(z_l) \rangle_\epsilon dy$$

and the convergence is in $C(O_\delta(\mathbf{z}))$. Differentiating and using (3.9) we obtain

$$\begin{aligned} \partial_z \mathcal{F}_{2,\epsilon}(z) &= \int \partial_z \frac{1}{(z - y)_\epsilon} \langle V_{-\frac{\gamma}{2},\epsilon}(z) V_{\gamma,\epsilon}(y) \prod_l V_{\alpha_l,\epsilon}(z_l) \rangle_\epsilon dy \\ &+ \frac{\gamma^2}{4} \int \left(\frac{1}{(z - y)_\epsilon} \right)^2 \langle V_{-\frac{\gamma}{2},\epsilon}(z) V_{\gamma,\epsilon}(y) \prod_l V_{\alpha_l,\epsilon}(z_l) \rangle_\epsilon dy \\ &+ \frac{\gamma}{4} \sum_i \frac{\alpha_i}{z - z_i} \int \frac{1}{(z - y)_\epsilon} \langle V_{-\frac{\gamma}{2},\epsilon}(z) V_{\gamma,\epsilon}(y) \prod_l V_{\alpha_l,\epsilon}(z_l) \rangle_\epsilon dy \\ &- \frac{\mu \gamma^2}{4} \int \int \frac{1}{(z - y)_\epsilon} \frac{1}{(z - x)_\epsilon} \langle V_{-\frac{\gamma}{2},\epsilon}(z) V_{\gamma,\epsilon}(y) V_{\gamma,\epsilon}(x) \prod_l V_{\alpha_l,\epsilon}(z_l) \rangle_\epsilon dy dx. \end{aligned}$$

In the same way as in Lemmas 3.3 and 3.4 we get

$$\lim_{\epsilon \rightarrow 0} \partial_z \mathcal{F}_{2,\epsilon}(z) = \pi \bar{A}(z) + \pi \bar{C}(z) + \frac{\gamma}{4} \sum_i \frac{\alpha_i}{z - z_i} \int \frac{1}{z - y} \langle V_{-\frac{\gamma}{2}}(z) V_\gamma(y) \prod_l V_{\alpha_l}(z_l) \rangle dy$$

where the limit is in $\mathcal{D}'(\mathbb{C} \setminus \{z_1, \dots, z_n\})$. In conclusion, we get the following in $\mathcal{D}'(\mathbb{C} \setminus \{z_1, \dots, z_n\})$:

$$\begin{aligned} & \frac{4}{\gamma^2} \partial_{zz}^2 \langle V_{-\frac{\gamma}{2}}(z) \prod_l V_{\alpha_l}(z_l) \rangle \\ &= -\frac{1}{\gamma} \sum_i \frac{\alpha_i}{(z - z_i)^2} \langle V_{-\frac{\gamma}{2}}(z) \prod_l V_{\alpha_l}(z_l) \rangle + \frac{1}{4} \sum_{i,j} \frac{\alpha_i \alpha_j}{(z - z_i)(z - z_j)} \langle V_{-\frac{\gamma}{2}}(z) \prod_l V_{\alpha_l}(z_l) \rangle \\ & - \frac{\mu\gamma}{2} \sum_i \frac{\alpha_i}{z - z_i} \int \frac{1}{z - y} \langle V_{-\frac{\gamma}{2}}(z) V_\gamma(y) \prod_l V_{\alpha_l}(z_l) \rangle dy - \mu\pi(\bar{A}(z) + \bar{C}(z)) \end{aligned}$$

Now using $-\mu\pi(\bar{A}(z) + \bar{C}(z)) = \mu\pi\bar{B}(z)$ and recalling (3.22) and (3.5) and collecting terms the BPZ equation holds in the sense of distributions. In fact, it holds in the strong sense, i.e. the correlations are real analytic because taking the $\partial_{\bar{z}\bar{z}}^2$ derivative of the equation is a hypoelliptic PDE. \square

3.7 Proof of theorem 2.3

The following expression for the correlation functions follows from (2.8) by getting rid of the vertex operators using Girsanov theorem and then doing the c -integral (see [8]):

$$\langle \prod_l V_{\alpha_l}(z_l) \rangle = e^{\frac{b}{2} \sum_i \alpha_i^2} e^{\frac{1}{2} \sum_{j \neq k} \alpha_j \alpha_k G(z_j, z_k)} \left(\prod_l \hat{g}(z_l)^{\Delta_{\alpha_l}} \right) \mu^{-s} \gamma^{-1} \Gamma(s) \mathbb{E}(Z_0^{-s}) \quad (3.26)$$

where $s = \frac{\sum_l \alpha_l - 2Q}{\gamma}$, and $b = \ln 2 - \frac{1}{2}$ is the additive constant in the covariance (2.1) and

$$Z_0 = e^{\frac{b\gamma^2}{2}} \int_{\mathbb{C}} e^{\gamma X(x) - \frac{\gamma^2}{2} \mathbb{E}[X(x)^2] + \gamma \sum_l \alpha_l G(x, z_l)} \hat{g}(x) dx.$$

Using the formula (2.1) we can write this as

$$\langle \prod_l V_{\alpha_l}(z_l) \rangle = A(\alpha) \prod_{j < k} \frac{1}{|z_j - z_k|^{\alpha_j \alpha_k}} \mu^{-s} \gamma^{-1} \Gamma(s) \mathbb{E}(Z_1^{-s}) \quad (3.27)$$

where

$$\begin{aligned} Z_1 &= e^{\frac{b\gamma^2}{2}} \int_{\mathbb{C}} e^{\gamma X(x) - \frac{\gamma^2}{2} \mathbb{E}[X(x)^2]} \prod_l \frac{1}{|x - z_l|^{\gamma \alpha_l}} \hat{g}(x)^{1 - \frac{\gamma}{4} \sum_l \alpha_l} dx. \\ A(\alpha) &= e^{b(2Q \sum \alpha_i - \frac{1}{2} (\sum \alpha_i)^2)}. \end{aligned}$$

Combining this with (2.14) we get the following expression for the three point structure constant:

$$C(\alpha_1, \alpha_2, \alpha_3) = A(\alpha_1, \alpha_2, \alpha_3) \mu^{-s} \gamma^{-1} \Gamma(s) \mathbb{E}(\rho(\alpha_1, \alpha_2, \alpha_3)^{-s})$$

where

$$\rho(\alpha_1, \alpha_2, \alpha_3) = e^{\frac{b\gamma^2}{2}} \int_{\mathbb{C}} e^{\gamma X(x) - \frac{\gamma^2}{2} \mathbb{E}[X(x)^2]} \frac{1}{|x|^{\gamma \alpha_1} |x - 1|^{\gamma \alpha_2}} \hat{g}(x)^{1 - \frac{\gamma}{4} \sum_l \alpha_l} dx.$$

Similarly, the function G defined in (2.15) becomes

$$G(z) = |z|^{\frac{\gamma \alpha_1}{2}} |z - 1|^{\frac{\gamma \alpha_2}{2}} \mathcal{T}(z)$$

where $\mathcal{T}(z)$ is given by

$$\mathcal{T}(z) = A(-\frac{\gamma}{2}, \alpha_1, \alpha_2, \alpha_3) \mu^{-s+\frac{1}{2}} \gamma^{-1} \Gamma(s - \frac{1}{2}) \mathbb{E}[R(z)^{\frac{1}{2}-s}] \quad (3.28)$$

and

$$R(z) = e^{\frac{b\gamma^2}{2}} \int_{\mathbb{C}} e^{\gamma X(x) - \frac{\gamma^2}{2} \mathbb{E}[X(x)^2]} \frac{|x-z|^{\frac{\gamma^2}{2}}}{|x|^{\gamma\alpha_1} |x-1|^{\gamma\alpha_2}} \hat{g}(x)^{1+\frac{\gamma^2}{8}-\frac{\gamma}{4} \sum_l \alpha_l} dx.$$

Note that

$$\mathcal{T}(0) = C(\alpha_1 - \frac{\gamma}{2}, \alpha_2, \alpha_3). \quad (3.29)$$

If we apply the BPZ equation to the expression (2.13) we get the following equation for G

$$\frac{4}{\gamma^2} \partial_{zz}^2 G(z) - (\frac{1}{z} + \frac{1}{z-1}) \partial_z G(z) + \left(\frac{\Delta_1}{z^2} + \frac{\Delta_2}{(z-1)^2} + \frac{\Delta_3 - \Delta_2 - \Delta_1 - \Delta_{-\frac{\gamma}{2}}}{z(z-1)} \right) G(z) = 0.$$

Obviously, the function \mathcal{T} is a positive and continuous and, after a bit of calculus, we see that the function \mathcal{T} is a solution of the hypergeometric equation

$$z(1-z) \partial_{zz}^2 \mathcal{T}(z) + (c - z(a+b+1)) \partial_z \mathcal{T}(z) - ab \mathcal{T}(z) = 0 \quad (3.30)$$

where a, b, c are given by (2.16) and (2.17). From Lemma 5.2 in the appendix, we know that around $z = 0$ the function \mathcal{T} is given by

$$\mathcal{T}(z) = \lambda_1 |F_-(z)|^2 + \lambda_2 \text{Re}(F_-(z) F_+(z)) + \lambda_3 \text{Im}(F_-(z) F_+(z)) + \lambda_4 |F_+(z)|^2$$

where $\lambda_i \in \mathbb{R}$ for all i . In order to conclude, we must show that $\lambda_1 = C(\alpha_1 - \frac{\gamma}{2}, \alpha_2, \alpha_3)$, $\lambda_2 = \lambda_3 = 0$ and $\lambda_4 = -\mu \frac{\pi}{l(-\frac{\gamma^2}{4})l(\frac{\gamma\alpha_1}{2})l(2+\frac{\gamma^2}{4}-\frac{\gamma\alpha_1}{2})} C(\alpha_1 + \frac{\gamma}{2}, \alpha_2, \alpha_3)$. Now recall that we have the following expansion as z goes to 0

$$\begin{aligned} & \lambda_1 |F_-(z)|^2 + \lambda_2 \text{Re}(F_-(z) F_+(z)) + \lambda_3 \text{Im}(F_-(z) F_+(z)) + \lambda_4 |F_+(z)|^2 \\ &= \lambda_1 + \lambda_2 \text{Re}(z^{1-c}) + \lambda_3 \text{Im}(z^{1-c}) + \lambda_4 |z|^{2(1-c)} + o(|z|^{2(1-c)}). \end{aligned}$$

We can conclude with the following lemma:

Lemma 3.6. *The function $\mathcal{T}(z)$ has the following expansion around $z = 0$*

$$\mathcal{T}(z) = C(\alpha_1 - \frac{\gamma}{2}, \alpha_2, \alpha_3) - \mu \frac{\pi}{l(-\frac{\gamma^2}{4})l(\frac{\gamma\alpha_1}{2})l(2+\frac{\gamma^2}{4}-\frac{\gamma\alpha_1}{2})} C(\alpha_1 + \frac{\gamma}{2}, \alpha_2, \alpha_3) |z|^{2(1-c)} + o(|z|^{2(1-c)}). \quad (3.31)$$

Proof. We have

$$\mathbb{E}[R(z)^{\frac{1}{2}-s}] - \mathbb{E}[R(0)^{\frac{1}{2}-s}] = (\frac{1}{2} - s) \int_0^1 \mathbb{E}[(R(z) - R(0)) R_t(z)^{-\frac{1}{2}-s}] dt$$

where $R_t(z) = tR(0) + (1-t)R(z)$. We set

$$m_z(u) = \frac{|u-z|^{\frac{\gamma^2}{2}} - |u|^{\frac{\gamma^2}{2}}}{|u|^{\gamma\alpha_1} |u-1|^{\gamma\alpha_2}} \hat{g}(u)^{1+\frac{\gamma^2}{8}-\frac{\gamma}{4} \sum_l \alpha_l}$$

and

$$R(z, u) = e^{\frac{b\gamma^2}{2}} \int_{\mathbb{R}^2} e^{\gamma X(x) - \frac{\gamma^2}{2} \mathbb{E}[X(x)^2]} \frac{|x-z|^{\frac{\gamma^2}{2}}}{|x|^{\gamma\alpha_1} |x-1|^{\gamma\alpha_2}} e^{\gamma^2 G(u, x)} \hat{g}(x)^{1+\frac{\gamma^2}{8}-\frac{\gamma}{4} \sum_l \alpha_l} dx.$$

Then, we have by Girsanov

$$\mathbb{E}[(R(z) - R(0)) R_t(z)^{-\frac{1}{2}-s}] = e^{\frac{b\gamma^2}{2}} \int_{\mathbb{C}} m_z(u) \mathbb{E}[R_t(z, u)^{-\frac{1}{2}-s}] du \quad (3.32)$$

where $R_t(z, u) = tR(0, u) + (1-t)R(z, u)$. Since the Green function G is bounded from below, we have the estimate

$$\inf_{|z| \leq \frac{1}{2}, u \in \mathbb{C}} R(z, u) \geq C \int_{2 \leq |x| \leq 3} e^{\gamma X(x) - \frac{\gamma^2}{2} \mathbb{E}[X(x)^2]} dx := W$$

so that

$$\sup_u \mathbb{E}[R_t(z, u)^{-\frac{1}{2}-s}] \leq C \mathbb{E}[W^{-\frac{1}{2}-s}] < \infty. \quad (3.33)$$

Hence

$$\int_{\mathbb{C}} m_z(u) 1_{|u| \geq \frac{1}{2}} \mathbb{E}[R_t(z, u)^{-\frac{1}{2}-s}] du \leq C \int_{\mathbb{R}^2} |m_z(u)| 1_{|u| \geq \frac{1}{2}} du \leq C|z|. \quad (3.34)$$

Therefore, in the equality (3.32) and up to a $O(|z|)$ term, we can put the indicator $1_{|u| \leq \frac{1}{2}}$ in the integral. By the change of variables $u = |z|v$ we then get

$$\int_{\mathbb{C}} m_z(u) 1_{|u| \leq \frac{1}{2}} \mathbb{E}[R_t(z, u)^{-\frac{1}{2}-s}] du = |z|^{2+\frac{\gamma^2}{2}-\gamma\alpha_1} \int_{\mathbb{C}} n_z(v) 1_{|v| \leq \frac{1}{2|z|}} \mathbb{E}[R_t(z, \frac{v}{|z|})^{-\frac{1}{2}-s}] dv$$

with

$$n_z(v) := \frac{|v - \frac{z}{|z|}|^{\frac{\gamma^2}{2}} - |v|^{\frac{\gamma^2}{2}}}{|v|^{\gamma\alpha_1} |1 - v|z||^{\gamma\alpha_2}} \hat{g}(|z|v)^{1+\frac{\gamma^2}{8}-\frac{\gamma}{4}\sum_l \alpha_l}.$$

We have then

$$n_z(v) 1_{|v| \leq \frac{1}{2|z|}} \leq C \sup_{|w|=1} \frac{||v-w|^{\frac{\gamma^2}{2}} - |v|^{\frac{\gamma^2}{2}}|}{|v|^{\gamma\alpha_1}} \equiv k(v)$$

Let $Q - \frac{1}{\gamma} < \alpha_1 < Q - \frac{\gamma}{2}$. Then $k \in L^1(\mathbb{C})$. Combining this with (3.33) implies that we may apply dominated convergence to conclude

$$\lim_{r \rightarrow 0} \int_{\mathbb{C}} n_{re^{i\theta}}(v) 1_{|v| \leq \frac{1}{2r}} \mathbb{E}[R_t(re^{i\theta}, \frac{v}{r})^{-\frac{1}{2}-s}] dv = \int_{\mathbb{C}} \frac{|v - e^{i\theta}|^{\frac{\gamma^2}{2}} - |v|^{\frac{\gamma^2}{2}}}{|v|^{\gamma\alpha_1}} \lim_{r \rightarrow 0} \mathbb{E}[R_t(re^{i\theta}, \frac{v}{r})^{-\frac{1}{2}-s}] dv$$

Also, the following convergence holds almost surely

$$R_t(re^{i\theta}, \frac{v}{r}) \xrightarrow{r \rightarrow 0} e^{\gamma^2 b} e^{\frac{b\gamma^2}{2}} \int_{\mathbb{C}} e^{\gamma X(x) - \frac{\gamma^2}{2} \mathbb{E}[X(x)^2]} \frac{1}{|x|^{\gamma(\alpha_1 + \frac{\gamma}{2})} |x-1|^{\gamma\alpha_2}} \hat{g}(x)^{1-\frac{\gamma^2}{8}-\frac{\gamma}{4}\sum_l \alpha_l} dx.$$

We conclude that

$$\mathbb{E}(R(z))^{\frac{1}{2}-s} = \mathbb{E}(R(0))^{\frac{1}{2}-s} + (\frac{1}{2} - s)|z|^{2(1-c)} e^{-\gamma^2 bs} \rho(\alpha_1 + \frac{\gamma}{2}, \alpha_2, \alpha_3) \int_{\mathbb{C}} \frac{|v-1|^{\frac{\gamma^2}{2}} - |v|^{\frac{\gamma^2}{2}}}{|v|^{\gamma\alpha_1}} dv + o(|z|^{2(1-c)}).$$

Combining this with (3.28) and (3.29) we obtain the claim (3.31) since

$$e^{-\gamma^2 bs} \frac{A(-\frac{\gamma}{2}, \alpha_1, \alpha_2, \alpha_3)}{A(\alpha_1 + \frac{\gamma}{2}, \alpha_2, \alpha_3)} = 1$$

and

$$\int_{\mathbb{C}} \frac{|v-1|^{\frac{\gamma^2}{2}} - |v|^{\frac{\gamma^2}{2}}}{|v|^{\gamma\alpha_1}} dv = \frac{\pi}{l(-\frac{\gamma^2}{4})l(\frac{\gamma\alpha_1}{2})l(2 + \frac{\gamma^2}{4} - \frac{\gamma\alpha_1}{2})}$$

where we used an analytic continuation of (5.12) in the appendix. \square

3.8 Proof of corollary 2.4

We will use a standard formula on the hypergeometric functions to prove the corollary. The idea is to exploit the fact that we have two different expressions of the function G : one that corresponds to an asymptotic expansion around 0 and one that corresponds to an asymptotic expansion around 1. This procedure corresponds in the physics literature to the crossing symmetry condition for conformal blocks: see [24]. Along the same lines as the proof of theorem 2.3, one can show that around $z = 1$ the function G is of the form

$$\frac{G(z)}{|z|^{\frac{\gamma\alpha_1}{2}}|z-1|^{\frac{\gamma\alpha_2}{2}}} = C(\alpha_1, \alpha_2 - \frac{\gamma}{2}, \alpha_3)|\widetilde{F}_-(z)|^2 - \mu \frac{\pi}{l(-\frac{\gamma^2}{4})l(\frac{\gamma\alpha_2}{2})l(2 + \frac{\gamma^2}{4} - \frac{\gamma\alpha_2}{2})} C(\alpha_1, \alpha_2 + \frac{\gamma}{2}, \alpha_3)|\widetilde{F}_+(z)|^2 \quad (3.35)$$

where we have

$$\widetilde{F}_-(z) = {}_2F_1(a, b, 1 + a + b - c, 1 - z)$$

and

$$\widetilde{F}_+(z) = (1 - z)^{c-a-b} {}_2F_1(c - a, c - b, 1 + c - a - b, 1 - z).$$

Recall the following standard formulas on hypergeometric functions for $x \in]0, 1[$ (see [24])

$$F_-(z) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \widetilde{F}_-(x) + \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} \widetilde{F}_+(x)$$

and

$$F_+(z) = \frac{\Gamma(2-c)\Gamma(c-a-b)}{\Gamma(1-a)\Gamma(1-b)} \widetilde{F}_-(x) + \frac{\Gamma(2-c)\Gamma(a+b-c)}{\Gamma(a-c+1)\Gamma(b-c+1)} \widetilde{F}_+(x).$$

If we expand expression (2.3) using the two previous formulas in the $(\widetilde{F}_-(x), \widetilde{F}_+(x))$ basis, we get

$$\frac{G(x)}{|x|^{\frac{\gamma\alpha_1}{2}}|x-1|^{\frac{\gamma\alpha_2}{2}}} = C_1|\widetilde{F}_-(x)|^2 + C_2|\widetilde{F}_+(x)|^2 + C_3(\overline{\widetilde{F}_-(x)}\widetilde{F}_-(x) + \widetilde{F}_-(x)\overline{\widetilde{F}_-(x)})$$

where C_3 has the following expression

$$C_3 = C(\alpha_1 - \frac{\gamma}{2}, \alpha_2, \alpha_3) \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} - C(\alpha_1 + \frac{\gamma}{2}, \alpha_2, \alpha_3) \frac{\mu\pi}{l(-\frac{\gamma^2}{4})l(\frac{\gamma\alpha_1}{2})l(2 + \frac{\gamma^2}{4} - \frac{\gamma\alpha_1}{2})} \frac{\Gamma(2-c)\Gamma(c-a-b)}{\Gamma(1-a)\Gamma(1-b)} \frac{\Gamma(2-c)\Gamma(a+b-c)}{\Gamma(a-c+1)\Gamma(b-c+1)}.$$

Now, expression (2.3) and expression (3.35) must coincide on the segment $]0, 1[$ and therefore the constant C_3 must be equal to 0. Therefore, we get the following expression By equation G with both expressions and using the above change of base, we get that

$$\frac{C(\alpha_1 + \frac{\gamma}{2}, \alpha_2, \alpha_3)}{C(\alpha_1 - \frac{\gamma}{2}, \alpha_2, \alpha_3)} = \frac{l(-\frac{\gamma^2}{4})l(\frac{\gamma\alpha_1}{2})l(2 + \frac{\gamma^2}{4} - \frac{\gamma\alpha_1}{2})}{\mu\pi} \frac{\Gamma(c)^2}{\Gamma(2-c)^2} \frac{1}{l(a)l(b)l(c-a)l(c-b)}.$$

Now, we have

$$a = \frac{\gamma}{4}(\alpha_1 + \alpha_2 + \alpha_3 - 2Q - \frac{\gamma}{2})$$

and

$$c - a = 1 - \frac{\gamma}{4}(-\alpha_1 + \alpha_2 + \alpha_3 - \frac{\gamma}{2}), \quad \frac{1}{l(c-a)} = l(\frac{\gamma}{4}(-\alpha_1 + \alpha_2 + \alpha_3 - \frac{\gamma}{2})),$$

where we have used $\frac{1}{l(1-x)} = l(x)$. Also

$$c - b = \frac{\gamma}{4}(\alpha_1 - \alpha_2 + \alpha_3 - \frac{\gamma}{2}), \quad b = \frac{\gamma}{4}(\alpha_1 + \alpha_2 + \alpha_3 - \frac{\gamma}{2}).$$

Finally

$$l(2 + \frac{\gamma^2}{4} - \frac{\gamma\alpha_1}{2}) \frac{\Gamma(c)^2}{\Gamma(2-c)^2} = \frac{\Gamma(2-c)}{\Gamma(c-1)} \frac{\Gamma(c)^2}{\Gamma(2-c)^2} = (c-1) \frac{\Gamma(c)}{\Gamma(2-c)} = -l(c) = -l(\frac{\gamma\alpha_1}{2} - \frac{\gamma^2}{4}).$$

These identities give the expression for the constant of Corollary 2.4. \square

4 Estimates on the correlation functions

In this section, we study the regularity and blowing up of the correlation functions when merging 2 or more insertion points.

4.1 L^p estimates

In the whole section on L^p estimates, X_ϵ stands for the ϵ -regularization of the Free Field in terms of circle average or mollification (it does not matter). Recall that we need to bound the correlations (3.1) and (3.2) where the points \mathbf{z} are fixed, in a bounded region and non coinciding, but x and y may get together or close to one of the z_i . Also, we need to get decay as x or y goes to infinity. We fix a constant $\delta > 0$ so that $\delta > 0$ will measure the minimal distance between the points not getting together and δ^{-1} minimal distance from origin of the points going to infinity. The constants C in the bounds will be δ dependent and all the weights α of the vertex operators satisfy $\alpha < Q$.

First let us consider correlation functions when two points x_1 and x_2 get together in a ball of radius δ^{-1} (these could be x, z_i in (3.1) or x, y in (3.2) or x, z_i in (3.2) or y, z_i in (3.2)). Define

$$U_{N,\delta} = \{\mathbf{z} \in \mathbb{C}^N \mid \min_{i \neq j} |z_i - z_j| \geq \delta\}$$

and recall the set (3.3). Set

$$|z|_\epsilon := \epsilon \vee |z|.$$

Then we have:

Proposition 4.1. *Let $\mathbf{z} \in U_{N,\delta}$ and $x_1, x_2 \in O_\delta(\mathbf{z})$ with $x_1, x_2, z_2, \dots, z_N \in B(0, \delta^{-1})$. Let the weights $\beta_1, \beta_2, \alpha_i$ satisfy*

$$\beta_1 + \beta_2 \geq Q \text{ and } \sum \alpha_i > Q \text{ or } \beta_1 + \beta_2 < Q \text{ and } \beta_1 + \beta_2 + \sum \alpha_i > 2Q.$$

Then

$$\langle V_{\beta_1, \epsilon}(x_1) V_{\beta_2, \epsilon}(x_2) \prod_k V_{\alpha_k, \epsilon}(z_k) \rangle_\epsilon \leq C(\delta) \hat{g}(z_1)^{\Delta_{\alpha_1}} |x_1 - x_2|_\epsilon^{2\Delta_{(\beta_1 + \beta_2) \wedge Q} - 2\Delta_{\beta_1} - 2\Delta_{\beta_2}} |\ln(|x_1 - x_2|_\epsilon)|^{-c_{\beta_1, \beta_2}}$$

where $c_{\beta_1, \beta_2} = \frac{3}{2}$ if $\beta_1 + \beta_2 > Q$, $c_{\beta_1, \beta_2} = \frac{1}{2}$ if $\beta_1 + \beta_2 = Q$ and $c_{\beta_1, \beta_2} = 0$ if $\beta_1 + \beta_2 < Q$.

Remark 4.2. *This estimate can be thought of as a "fusion rule" $V_{\beta_1} \times V_{\beta_2} = V_{(\beta_1 + \beta_2) \wedge Q}$ up to logarithmic corrections.*

Next we merge three vertex operators. Note that we need this only when all the insertions are in a bounded region. The next proposition reflects the above fusion rules: we first fuse the closest pair of points and then the third point is fused to the result.

Proposition 4.3. (merging of three vertex operators) *Assume that $x_1, x_2, x_3 \in O_\delta(\mathbf{z}) \cap B(0, \delta^{-1})$ and let $|x_1 - x_2| \leq |x_1 - x_3| \leq |x_2 - x_3|$. Then*

(a) *If $\sum_i \alpha_i > Q$, $\beta_1 + \beta_2 \geq Q$ and $\beta_1 + \beta_2 + \beta_3 \geq Q$ and $\beta_3 \geq 0$ then*

$$\langle V_{\beta_1, \epsilon}(x_1) V_{\beta_2, \epsilon}(x_2) V_{\beta_3, \epsilon}(x_3) \prod_k V_{\alpha_k, \epsilon}(z_k) \rangle_\epsilon \leq C(\delta) |x_1 - x_2|_\epsilon^{\frac{Q^2}{2} - 2\Delta_{\beta_1} - 2\Delta_{\beta_2}} |x_1 - x_3|_\epsilon^{-2\Delta_{\beta_3}}.$$

(b) If $\beta_3 + \sum_i \alpha_i > Q$, $\beta_1 + \beta_2 \geq Q$ and $\beta_3 < 0$ then

$$\langle V_{\beta_1, \epsilon}(x_1) V_{\beta_2, \epsilon}(x_2) V_{\beta_3, \epsilon}(x_3) \prod_k V_{\alpha_k, \epsilon}(z_k) \rangle_\epsilon \leq C(\delta) |x_1 - x_2|_\epsilon^{\frac{Q^2}{2} - 2\Delta_{\beta_1} - 2\Delta_{\beta_2}} |x_2 - x_3|_\epsilon^{-\beta_3 Q}.$$

(c) If $\sum_i \alpha_i > Q$, $\beta_1 + \beta_2 < Q$ and $\beta_1 + \beta_2 + \beta_3 \geq Q$ then

$$\langle V_{\beta_1, \epsilon}(x_1) V_{\beta_2, \epsilon}(x_2) V_{\beta_3, \epsilon}(x_3) \prod_k V_{\alpha_k, \epsilon}(z_k) \rangle_\epsilon \leq C(\delta) |x_1 - x_2|_\epsilon^{-\beta_1 \beta_2} |x_1 - x_3|_\epsilon^{\frac{1}{2}(\beta_1 + \beta_2 + \beta_3 - Q)^2 - \beta_1 \beta_3} |x_2 - x_3|_\epsilon^{-\beta_2 \beta_3}.$$

(d) If $\sum \beta_i + \sum_j \alpha_j > 2Q$, $\beta_1 + \beta_2 < Q$ and $\beta_1 + \beta_2 + \beta_3 < Q$ then

$$\langle V_{\beta_1, \epsilon}(x_1) V_{\beta_2, \epsilon}(x_2) V_{\beta_3, \epsilon}(x_3) \prod_k V_{\alpha_k, \epsilon}(z_k) \rangle_\epsilon \leq C(\delta) |x_1 - x_2|_\epsilon^{-\beta_1 \beta_2} |x_1 - x_3|_\epsilon^{-\beta_1 \beta_3} |x_2 - x_3|_\epsilon^{-\beta_2 \beta_3}.$$

Remark 4.4. Let us stress that in the above proposition we did not bother proving optimal bounds in the sense that we did not focus on the logarithmic corrections (in the case of the merging of three points). We believe that our method gives sharp bounds in the power gauge. Yet it would be quite interesting and technically more involved to prove a general statement concerning the fusion (merging) of insertions in LQFT and show that merging points in the correlation functions gives (after renormalization) a correlation function with a new vertex operator at the place of fusion with weight given by the sum of the merging weights in case when their sum is strictly less than Q and Q otherwise. Let us further stress that this question of fusion is especially interesting mathematically when the resulting weights do not satisfy the Seiberg bound anymore (for instance consider four points $(z_i, \alpha_i)_{1 \leq i \leq 4}$ with respective weights $(\alpha_i)_i$ such that $\sum_i \alpha_i > 2Q$, $\alpha_1 + \alpha_2 > Q$, $\alpha_3 + \alpha_4 \leq Q$ and merge z_1, z_2).

Now we focus on the behaviour of the correlation functions close to ∞ . First, we have

Proposition 4.5. (Asymptotic behaviour near ∞) Let $N \geq 3$ and let the weights α satisfy $\sum_i \alpha_i > 2Q$. Let $\mathbf{z} \in U_{N, \delta}$ and $z_i \in B(0, \delta^{-1})$. Let $x \in B(0, \delta^{-1})^c \cap O_\delta(\mathbf{z})$. Then

$$\langle V_{\gamma, \epsilon}(x) \prod_k V_{\alpha_k, \epsilon}(z_k) \rangle_{\hat{g}, \epsilon} \leq C(\delta) |x|^{-4}.$$

Proposition 4.6. (merging of two vertex operators near ∞) Let $\mathbf{z} \in U_{N, \delta}$ and $z_i \in B(0, \delta^{-1})$. Let $x_1, x_2 \in B(0, \delta^{-1})^c \cap O_\delta(\mathbf{z})$. Then

1. if $\beta_1 + \beta_2 > Q$ and $\sum_i \alpha_i > Q$ then

$$\langle V_{\beta_1, \epsilon}(x_1) V_{\beta_2, \epsilon}(x_2) \prod_i V_{\alpha_i, \epsilon}(z_i) \rangle_\epsilon \leq C(\delta) |x_1|^{-4\Delta_{\beta_1}} |x_2|^{-4\Delta_{\beta_2}} \mathbf{1} \vee \left(\frac{|x_1 x_2|}{|x_1 - x_2|_\epsilon} \right)^{2\Delta_{\beta_1} + 2\Delta_{\beta_2} - \frac{1}{2}Q^2}.$$

2. if $\beta_1 + \beta_2 = Q$ and $\sum_i \alpha_i > Q$ then

$$\langle V_{\beta_1, \epsilon}(x_1) V_{\beta_2, \epsilon}(x_2) \prod_k V_{\alpha_k, \epsilon}(z_k) \rangle_\epsilon \leq C(\delta) |x_1|^{-4\Delta_{\beta_1} + \beta_1 \beta_2} |x_2|^{-4\Delta_{\beta_2} + \beta_1 \beta_2} |x_1 - x_2|_\epsilon^{-\beta_1 \beta_2}.$$

3. if $\beta_1 + \beta_2 < Q$ and $\beta_1 + \beta_2 + \sum_i \alpha_i > 2Q$ then

$$\langle V_{\beta_1, \epsilon}(x_1) V_{\beta_2, \epsilon}(x_2) \prod_k V_{\alpha_k, \epsilon}(z_k) \rangle_\epsilon \leq C(\delta) |x_1|^{-4\Delta_{\beta_1}} |x_2|^{-4\Delta_{\beta_2}} \mathbf{1} \vee \left(\frac{|x_1 x_2|}{|x_1 - x_2|_\epsilon} \right)^{\beta \theta}.$$

4.2 Proof of Proposition 3.1

Let us start with $f_{\epsilon, \mathbf{z}}$. From Proposition 4.1 we see that $\|f_{\epsilon}\|_{\infty} < \infty$ if $\epsilon > 0$. Integrability at infinity follows from Proposition 4.1 once we note $\Delta_{\gamma} = 1$ and $\hat{g}(z_1) = \mathcal{O}(|z_1|^{-4})$. Consider next local integrability near an insertion α_i at z_i . We claim that

$$f_{\epsilon, \mathbf{z}}(x) \leq C|x - z_i|^{-2+\zeta} \quad (4.1)$$

with $\zeta > 0$. Let first $\gamma + \alpha_i \geq Q$. Then Proposition 4.1 gives $\zeta = \frac{1}{2}(Q - \alpha_i)^2 > 0$ since $\alpha_i < Q$. If $\gamma + \alpha_i \leq Q$ we use $\gamma\alpha_i \leq \gamma Q - \gamma^2 = 2 - \frac{1}{2}\gamma^2$ so (4.1) holds with $\zeta = \frac{1}{2}\gamma^2$.

Consider now $F_{\epsilon, \mathbf{z}}(x, y)$. The claim (c) follows from the analogue of the bound (4.1) with α_i replaced by γ . Similarly integrability in (3.5) in the region $x \in O(\mathbf{z}), y \in O(\mathbf{z})^c$ follows from Proposition 4.1. We are left with the case where both x and y are close to one insertion point and the case where both x and y are outside a large ball.

Let us start with the case when x, y are close to an insertion place, say z_1 . We consider only the case when $\alpha_1 > 0$ since the case $\alpha_1 < 0$ is less singular. By the symmetry in exchanging x and y we have three cases to consider when applying Proposition 4.3:

- $|x - y| \leq |x - z_1| \leq |y - z_1|$:

$$F_{\mathbf{z}}(x, y) \leq \begin{cases} C|x - y|^{\frac{Q^2}{2}-4}|x - z_1|^{-2\Delta_{\alpha_1}}, & \text{if } 2\gamma + \alpha_1 \geq Q \text{ and } 2\gamma \geq Q, \\ C|x - y|^{-\gamma^2}|x - z_1|^{\frac{1}{2}Q^2-2\Delta_{\alpha_1}-2}, & \text{if } 2\gamma + \alpha_1 \geq Q \text{ and } 2\gamma < Q, \\ C|x - y|^{-\gamma^2}|x - z_1|^{-2\alpha_1\gamma}, & \text{if } 2\gamma + \alpha_1 < Q \text{ and } 2\gamma < Q \end{cases}$$

To discuss local integrability, consider the first case. Suppose first $\Delta_{\alpha_1} < 1$. Then $|x - z_1|^{-2\Delta_{\alpha_1}}$ is locally integrable and since $Q^2 > 4$ we get that $F_{\mathbf{z}}(x, y)|x - y|^{-\xi}$ is locally integrable for some $\xi > 0$. If $\Delta_{\alpha_1} \geq 1$ then

$$\begin{aligned} & \int |x - y|^{\frac{Q^2}{2}-4-\xi}|x - z_1|^{-2\Delta_{\alpha_1}} 1_{|x-y| \leq |x-z_1|} 1_{|x-z_1|, |y-z_1| < 1} dx dy \leq \\ & \int |u|^{\frac{Q^2}{2}-2-\xi-2\Delta_{\alpha_1}} 1_{|u| < 2} du = \int |u|^{-2+\frac{1}{2}(Q-\alpha_i)^2-\xi} 1_{|u| < 2} du < \infty \end{aligned}$$

for some $\xi > 0$. For the two other cases the two factors are separately integrable.

- $|x - z_1| \leq |y - z_1| \leq |x - y|$:

$$F_{\mathbf{z}}(x, y) \leq \begin{cases} C|x - z_1|^{\frac{Q^2}{2}-2-2\Delta_{\alpha_1}}|y - z_1|^{-2}, & \text{if } 2\gamma + \alpha_1 \geq Q \text{ and } \gamma + \alpha_1 \geq Q, \\ C|x - z_1|^{-\gamma\alpha_1}|y - z_1|^{\frac{Q^2}{2}-2\Delta_{\alpha_1}-4+\gamma\alpha_1}, & \text{if } 2\gamma + \alpha_1 \geq Q \text{ and } \gamma + \alpha_1 < Q, \\ C|x - z_1|^{-\gamma\alpha_1}|y - z_1|^{-\gamma(\alpha_1+\gamma)}, & \text{if } 2\gamma + \alpha_1 < Q \text{ and } \gamma + \alpha_1 < Q. \end{cases}$$

For integrability use $\frac{Q^2}{2} - 2\Delta_{\alpha_1} = \frac{1}{2}(Q - \alpha_i)^2 > 0$ in the first case and $\gamma(\alpha_1 + \gamma) < \gamma(Q - \gamma) = 2 - \frac{1}{2}\gamma^2$ in the last case. For the second case, if $\frac{Q^2}{2} - 2\Delta_{\alpha_1} - 4 + \gamma\alpha_1 < -2$, integrating over that factor produces $|x - z_1|^{\frac{Q^2}{2}-2\Delta_{\alpha_1}-2}$ which is integrable.

- $|x - z_1| \leq |x - y| \leq |y - z_1|$:

$$F_{\mathbf{z}}(x, y) \leq \begin{cases} C|x - z_1|^{\frac{Q^2}{2}-2-2\Delta_{\alpha_1}}|x - y|^{-2}, & \text{if } 2\gamma + \alpha_1 \geq Q \text{ and } \gamma + \alpha_1 \geq Q, \\ C|x - z_1|^{-\gamma\alpha_1}|x - y|^{\frac{Q^2}{2}-4-2\Delta_{\alpha_1}+\gamma\alpha_1}, & \text{if } 2\gamma + \alpha_1 \geq Q, \gamma + \alpha_1 < Q, \\ C|x - z_1|^{-\alpha_1\gamma}|x - y|^{-\gamma(\gamma+\alpha_1)}, & \text{if } 2\gamma + \alpha_1 < Q \text{ and } \gamma + \alpha_1 < Q. \end{cases}$$

Local integrability is as in the previous case.

Finally, let $|x|, |y| > R$ for R large enough. We use Proposition 4.6 with $\beta_1 = \beta_2 = \gamma$. Since $\Delta_{\gamma} = 1$ it is readily seen that

$$F_{\mathbf{z}, \epsilon}(x, y) \leq C|xy|^{-2-\zeta}|x - y|_{\epsilon}^{-2+\eta}$$

for $\zeta, \eta > 0$ whereby the bounds in Proposition 3.1 follow. \square

4.3 Hölder type estimates

Now we turn to Hölder type estimates for the correlation functions. We define, for $\epsilon > 0$ the function

$$\mathcal{F}_\epsilon(y, z) = \langle V_{\gamma, \epsilon}(y) V_{-\frac{\gamma}{2}, \epsilon}(z) \prod_k V_{\alpha_i, \epsilon}(z_i) \rangle_\epsilon$$

By the definition (2.6)

$$\mathcal{F}_\epsilon(z, z) = (A\epsilon)^{\frac{\gamma^2}{2}} \langle V_{\frac{\gamma}{2}, \epsilon}(z) \prod_k V_{\alpha_i, \epsilon}(z_i) \rangle_\epsilon.$$

Let $\mathcal{F}(y, z) = \lim_{\epsilon \rightarrow 0} \mathcal{F}_\epsilon(y, z)$ which is defined and continuous in $(x, y) \in (\mathbb{C} \setminus \cup z_i)^2 \setminus D$ where D is the diagonal $\{(z, z) | z \in \mathbb{C}\}$ in \mathbb{C}^2 . By Proposition 4.1

$$|\mathcal{F}(y, z)| \leq C(\mathbf{z}) |y - z|^{\frac{\gamma^2}{2}} \quad (4.2)$$

and thus the function \mathcal{F} extends continuously to $(\mathbb{C} \setminus \cup z_i)^2$ by setting $\mathcal{F}(z, z) = \lim_{\epsilon \rightarrow 0} \mathcal{F}_\epsilon(z, z) = 0$.

We claim

Proposition 4.7. (Hölder type estimates) *For any $\delta > 0$, there exists a constant $C_\delta > 0$ s.t. for all $y, z \in B(0, \delta^{-1}) \cap O_\delta(\mathbf{z})$*

$$|\mathcal{F}_\epsilon(y, z) - \mathcal{F}_\epsilon(z, z)| \leq \begin{cases} C_\delta \epsilon^{\frac{\gamma^2-1}{2}} |y - z|^{1/2}, & \text{if } |y - z| \leq \epsilon \\ C_\delta |y - z|^{\frac{\gamma^2}{2}} & \text{if } |y - z| \geq \epsilon \end{cases}.$$

4.4 Applications to pointwise/distributional convergence of Beurling-type integrals involving correlation functions

Proof of Lemma 3.3.

The mapping $y \mapsto \frac{1}{(z-y)^2} \mathcal{F}(y, z)$ is obviously continuous on $\mathbb{C} \setminus \{z, z_1, \dots, z_N\}$. By Proposition 4.5, it is integrable near ∞ . Furthermore, it possesses singularities at the points $\{z, z_1, \dots, z_N\}$. Integrability near the $(z_i)_i$ follows from Proposition 4.1 and near z it follows from (4.2).

Let us show now that $z \mapsto \bar{A}_\epsilon(z)$ converges in $\mathcal{D}'(O_\delta(\mathbf{z}))$ for all $\delta > 0$. Let $\varphi \in C_0^\infty(O_\delta(\mathbf{z}))$. Recalling the definition of $\bar{A}_\epsilon(z)$, it is the sum of two terms (see (3.19)). The convergence of the second term is obvious so that we only treat the first one, call it $\bar{A}_\epsilon^1(z)$. We have

$$\int \varphi(z) \bar{A}_\epsilon^1(z) dz = - \int \varphi(z) \int \frac{1}{(z-y)^2} (\mathcal{F}_\epsilon(y, z) - \mathbf{1}_{|y-z| \leq 1} \mathcal{F}_\epsilon(z, z)) dy dz$$

By Proposition 4.7 (applied with a δ such that $\text{Supp}(\varphi) \subset B(0, \delta^{-1})$), we have

$$\int |\varphi(z) \bar{A}_\epsilon^1(z)| \mathbf{1}_{|y-z| < \epsilon} dz \leq C \epsilon^{\frac{\gamma^2-1}{2}} \int \mathbf{1}_{|y-z| < \epsilon} \frac{|\varphi(z)|}{|z-y|^{3/2}} dy dz \leq C \epsilon^{\frac{\gamma^2}{2}} \int |\varphi(z)| dz.$$

This quantity converges to 0 as $\epsilon \rightarrow 0$. Now we can use the second bound of Proposition 4.7 and apply the dominated convergence theorem to infer convergence of $\int \varphi(z) \bar{A}_\epsilon^1(z) \mathbf{1}_{|y-z| \geq \epsilon} dz$. \square

Proof of Lemma 3.4.

Let us first discuss the integrability of the mapping

$$(x, y) \mapsto \phi_\epsilon(x, y, z) := \frac{1}{(z-x)(z-y)} \langle V_{-\frac{\gamma}{2}, \epsilon}(z) V_{\gamma, \epsilon}(x) V_{\gamma, \epsilon}(y) \prod_i V_{\alpha_i, \epsilon}(z_i) \rangle_\epsilon.$$

If x, y belong to a ball $B(0, R)$ then integrability in a neighborhood of the diagonal D and away from the points $\{(w, w); w \in \{z, z_1, \dots, z_N\}\}$ results from Proposition 4.1. We let the reader check this point (distinguish the cases $2\gamma > Q$, $2\gamma = Q$ and $2\gamma < Q$). Integrability when x and/or y approach ∞ results from Propositions 4.5 and 4.6.

We are left with the case when both x and y get close to z . This situation is described by Proposition 4.3. We proceed as in the proof of Proposition 3.1 but this time paying attention to the fact that one weight is negative. There are once again two cases depending on $2\gamma \geq Q$ or $2\gamma < Q$. The most intricate is the case $2\gamma \geq Q$ and this is the only case that we will discuss. Proposition 4.3 thus gives:

- $|x - y| \leq |x - z| \leq |y - z|$:

$$|\phi_\epsilon(x, y, z)| \leq C|x - y|_\epsilon^{-4 + \frac{Q^2}{2}} \frac{|y - z|_\epsilon^{1 + \frac{\gamma^2}{4}}}{|x - z||y - z|}$$

- $|x - z| \leq |y - z| \leq |x - y|$:

$$|\phi_\epsilon(x, y, z)| \leq C \frac{|x - y|_\epsilon^{-\gamma^2}}{|x - z||y - z|} \begin{cases} |y - z|_\epsilon^{\gamma^2 + \frac{(Q - 3\gamma/2)^2}{2}}, & \text{if } 3\gamma/2 \geq Q, \\ |y - z|_\epsilon^{\gamma^2}, & \text{if } 3\gamma/2 < Q. \end{cases}$$

- $|x - z| \leq |x - y| \leq |y - z|$:

$$|\phi(x, y, z)| \leq \begin{cases} C \frac{|x - z|_\epsilon^{\frac{\gamma^2}{2}}}{|x - z|} |x - y|^{-\gamma^2 + \frac{(Q - 3\gamma/2)^2}{2}} \frac{(\epsilon + |y - z|)^{\frac{\gamma^2}{2}}}{|y - z|}, & \text{if } 3\gamma/2 \geq Q, \\ C \frac{|x - z|_\epsilon^{\frac{\gamma^2}{2}}}{|x - z|} (\epsilon + |x - y|)^{-\gamma^2} \frac{|y - z|_\epsilon^{\frac{\gamma^2}{2}}}{|y - z|}, & \text{if } 3\gamma/2 < Q. \end{cases}$$

In the same way, for the $-\frac{2}{\gamma}$ -insertion, i.e. when considering the mapping

$$(x, y) \mapsto \phi_\epsilon(x, y, z) := \frac{1}{(z - x)(z - y)} \langle V_{-\frac{2}{\gamma}, \epsilon}(z) V_{\gamma, \epsilon}(x) V_{\gamma, \epsilon}(y) \prod_i V_{\alpha_i, \epsilon}(z_i) \rangle_{\hat{g}, \epsilon},$$

we get the following bounds:

- $|x - y| \leq |x - z| \leq |y - z|$:

$$|\phi(x, y, z)| \leq C|x - y|_\epsilon^{-4 + \frac{Q^2}{2}} \frac{|x - z|_\epsilon^{\frac{4}{\gamma^2} - 1}}{|x - z|} \frac{|y - z|_\epsilon^2}{|y - z|},$$

- $|x - z| \leq |y - z| \leq |x - y|$:

$$|\phi(x, y, z)| \leq \begin{cases} C \frac{|x - z|_\epsilon^2}{|x - z|} \frac{|y - z|_\epsilon^{2 + \frac{1}{2}(Q - 2\gamma + \frac{2}{\gamma})^2}}{|y - z|} (|x - y|_\epsilon)^{-\gamma^2}, & \text{if } 2\gamma - \frac{2}{\gamma} \geq Q, \\ C \frac{|x - z|_\epsilon^2}{|x - z|} \frac{|y - z|_\epsilon^2}{|y - z|} |x - y|_\epsilon^{-\gamma^2}, & \text{if } 3\gamma/2 < Q. \end{cases}$$

- $|x - z| \leq |x - y| \leq |y - z|$:

$$|\phi(x, y, z)| \leq \begin{cases} C \frac{|x - z|_\epsilon^{\frac{\gamma^2}{2}}}{|x - z|} |x - y|_\epsilon^{-\gamma^2 + \frac{(Q - 2\gamma + \frac{2}{\gamma})^2}{2}} \frac{|y - z|_\epsilon^{\frac{\gamma^2}{2}}}{|y - z|}, & \text{if } 2\gamma - \frac{2}{\gamma} \geq Q, \\ C \frac{|x - z|_\epsilon^2}{|x - z|} |x - y|_\epsilon^{-\gamma^2} \frac{|y - z|_\epsilon^2}{|y - z|}, & \text{if } 2\gamma - \frac{2}{\gamma} < Q. \end{cases}$$

In both cases integrability follows.

4.5 Preliminary remarks

Regularized correlations

The regularized version of (3.27) reads

$$\langle \prod_l V_{\alpha_l, \epsilon}(z_l) \rangle_\epsilon = A(\alpha) \prod_{j < k} e^{\alpha_j \alpha_k G_\epsilon(z_j - z_k)} \mu^{-s} \gamma^{-1} \Gamma(s) \mathbb{E}(Z_\epsilon^{-s}) \quad (4.3)$$

where

$$Z_\epsilon = e^{\frac{b\gamma^2}{2}} \int_{\mathbb{C}} \prod_l e^{\gamma \alpha_l G_\epsilon(x - z_l)} \hat{g}_\epsilon(x)^{1 - \frac{\gamma}{4} \sum_l \alpha_l} M_\epsilon(dx), \quad (4.4)$$

$G_\epsilon = \rho_\epsilon * \ell * \rho_\epsilon$ with $\ell(z) = \ln|z|^{-1}$, $\ln \hat{g}_\epsilon = \rho_\epsilon * \ln \hat{g}$, $M_\epsilon(dx) = e^{\gamma X_\epsilon - \frac{\gamma^2}{2} \mathbb{E} X_\epsilon^2} dx$ is the regularized chaos measure and $s = (\sum_l \alpha_l - 2Q)/\gamma$. We have the estimate

$$\ln|x|_\epsilon^{-1} - C \leq G_\epsilon(x) \leq \ln|x|_\epsilon^{-1} + C$$

where C is uniform in ϵ and $|x|_\epsilon := |x| \vee \epsilon$. Moreover $\|\partial_x^n G_\epsilon\|_\infty < \infty$ if $\epsilon > 0$. From this we deduce the smoothness of the corrections if $\epsilon > 0$ claimed in Proposition 3.1 (a).

Radial decomposition of the chaos measure

Here we summarize some results described in [9]. We denote by \mathcal{F}_η ($\eta > 0$) the sigma-algebra generated by the field X "away from the disc $B(0, \eta)$ ", namely

$$\mathcal{F}_\eta = \sigma\{X(f); f \text{ smooth, } \text{supp}(f) \in B(0, \eta)^c\}. \quad (4.5)$$

\mathcal{F}_∞ stands for the sigma algebra generated by $\bigcup_{\eta > 0} \mathcal{F}_\eta$.

First recall the standard observation that, for all $\eta > 0$, the process

$$t \in \mathbb{R}_+ \mapsto X_{\eta e^{-t}}(0) - X_\eta(0)$$

evolves as a Brownian motion independent of the sigma algebra \mathcal{F}_η . The following radial decomposition of the field X will be useful for the analysis (this observation was already made in [12])

Lemma 4.8. *The field X may be decomposed (in the sense of distributions) as*

$$X(z) = X_{|z|}(0) + Y(z) \quad (4.6)$$

where the centered Gaussian field Y is independent of the process $r \in \mathbb{R}_+^* \mapsto X_r(0)$ and has the following covariance

$$\mathbb{E}[Y(re^{i\sigma})Y(r'e^{i\sigma'})] = \ln \frac{r \vee r'}{|re^{i\sigma} - r'e^{i\sigma'}|}.$$

Using the above lemma, we get the following decomposition of the chaos measure

$$M_\gamma(dz) = c_\gamma \hat{g}(z) |z|^{\frac{\gamma^2}{2}} e^{\gamma X_{\hat{g}, |z|}(0)} M_\gamma(dz, Y)$$

where $M_\gamma(dz, Y)$ is the multiplicative chaos measure of the field Y with respect to the Lebesgue measure (i.e. $\mathbb{E} M_\gamma(dz, Y) = dz$) and $c_\gamma := e^{\frac{\gamma^2}{2} b - \frac{\gamma^2}{2} \mathbb{E}[X_1(0)^2]}$ is some constant.

We will now make change of variables $z = e^{-s+i\sigma}$, $s \in \mathbb{R}_+$, $\sigma \in [0, 2\pi)$ and let $\mu_Y(ds, d\sigma)$ be the multiplicative chaos measure of the field $Y(e^{-s+i\sigma})$ with respect to the measure $ds d\sigma$. We will denote by x_s the process

$$s \in \mathbb{R}_+ \rightarrow x_s := X_{e^{-s}}(0).$$

We arrived at the following useful "radial" decomposition of the chaos measure

Lemma 4.9. For $\delta > 0$ and on the ball $B(0, \delta^{-1})$, we have the decomposition

$$\int_A M_\gamma(dx) = c_\gamma \int_0^\infty \int_0^{2\pi} \mathbf{1}_A(e^{-s} e^{i\sigma}) e^{\gamma x_s - \gamma Qs} \hat{g}(e^{-s}) \mu_Y(ds, d\sigma)$$

for all measurable set $A \subset B(0, \delta^{-1})$ where $\mu_Y(ds, d\sigma)$ is a measure independent of the whole process $(x_s)_{s \geq 0}$. Furthermore, for all $q \in]-\infty; \frac{4}{\gamma^2}[$, we have

$$\sup_{a>0} \mathbb{E} \left[\left(\int_a^{a+1} \int_0^{2\pi} e^{\gamma(x_s - x_a)} \mu_Y(ds, d\sigma) \right)^q \right] < +\infty. \quad (4.7)$$

On the supremum of a drifted Brownian motion

To begin with, we state the two following lemmas.

Lemma 4.10. Let B be a standard Brownian motion.

1) For $\beta, \alpha > 0$, we have

$$\mathbb{P}(\sup_{u \leq t} B_u + \alpha u \leq \beta) \leq \frac{e^{-\frac{\alpha^2 t}{2}}}{\alpha^2 t^{3/2}} \sqrt{\frac{2}{\pi}} \beta e^{\alpha \beta}.$$

2) For $\beta > 0$, we have

$$\mathbb{P}(\sup_{u \leq t} B_u \leq \beta) = \sqrt{\frac{2}{\pi}} \int_0^{\frac{\beta}{\sqrt{t}}} e^{-\frac{u^2}{2}} du \leq \sqrt{\frac{2}{\pi}} \frac{\beta}{\sqrt{t}}.$$

The proof of this lemma is elementary so that the proof is left to the reader.

Now we fix some constants $\alpha, \tilde{\alpha} \in \mathbb{R}$ and set for some $s > 0$

$$\forall u \geq 0, \quad F(u) = \alpha u + \tilde{\alpha} \int_0^u \mathbf{1}_{[0,s]}(v) dv = \alpha u + \tilde{\alpha} u \wedge s.$$

Lemma 4.11. Define

$$\forall u \geq 0, \quad y_u = B_u + F(u).$$

Then we have the following estimates for $\beta \geq 0$ and $s < r$:

1) if $\alpha \geq 0, \tilde{\alpha} \geq 0$ then

$$\mathbb{P}(\sup_{u \in [0,r]} B_u + F(u) \leq \beta) \leq e^{-\frac{(\alpha+\tilde{\alpha})^2}{2}s - \frac{\alpha^2}{2}(r-s)} e^{(\alpha+\tilde{\alpha})\beta}.$$

2) if $\alpha \geq 0, \tilde{\alpha} \leq 0$ then

$$\mathbb{P}(\sup_{u \in [0,r]} B_u + F(u) \leq \beta) \leq e^{-\frac{(\alpha+\tilde{\alpha})^2 - \alpha^2 - \tilde{\alpha}^2}{2}s - \frac{\alpha^2}{2}r} e^{\alpha\beta}.$$

Proof. Let us set $f(u) = \alpha + \tilde{\alpha} \mathbf{1}_{[0,s]}(u)$. Using the Girsanov transform yields

$$\begin{aligned} \mathbb{P}(\sup_{u \in [0,wr]} B_u + F(u) \leq \beta) &= \mathbb{E} \left[e^{\int_0^r f(u) dB_u - \frac{1}{2} \int_0^r f(u)^2 du} \mathbf{1}_{\{\sup_{u \in [0,r]} B_u \leq \beta\}} \right] \\ &= e^{-\frac{(\alpha+\tilde{\alpha})^2}{2}s - \frac{\alpha^2}{2}(r-s)} \mathbb{E} \left[e^{\alpha B_r + \tilde{\alpha} B_s} \mathbf{1}_{\{\sup_{u \in [0,r]} B_u \leq \beta\}} \right]. \end{aligned} \quad (4.8)$$

From this formula, one can obtain the various items listed above:

- in the case of item 1, we have $\alpha \geq 0, \tilde{\alpha} \geq 0$. Hence we deduce by using $B_r \leq \beta, B_s \leq \beta$

$$\mathbb{P}(\sup_{u \in [0,wr]} B_u + F(u) \leq \beta) \leq e^{-\frac{(\alpha+\tilde{\alpha})^2}{2}s - \frac{\alpha^2}{2}(r-s)} e^{(\alpha+\tilde{\alpha})\beta}.$$

- concerning item 2, we plug the following estimate into (4.8)

$$\begin{aligned}
\mathbb{E} \left[e^{\alpha B_r + \tilde{\alpha} B_s} \mathbf{1}_{\{\sup_{u \in [0, r]} B_u \leq \beta\}} \right] &\leq e^{\alpha \beta} \mathbb{E} \left[e^{\tilde{\alpha} B_s} \mathbf{1}_{\{\sup_{u \in [0, s]} B_u \leq \beta\}} \right] \\
&= e^{\alpha \beta + \frac{\tilde{\alpha}^2}{2} s} \mathbb{E} \left[e^{\tilde{\alpha} B_s - \frac{\tilde{\alpha}^2}{2} s} \mathbf{1}_{\{\sup_{u \in [0, s]} B_u \leq \beta\}} \right] \\
&= e^{\alpha \beta + \frac{\tilde{\alpha}^2}{2} s} \mathbb{E} \left[\mathbf{1}_{\{\sup_{u \in [0, s]} B_u + \tilde{\alpha} u \leq \beta\}} \right] \\
&\leq e^{\alpha \beta + \frac{\tilde{\alpha}^2}{2} s}. \quad \square
\end{aligned}$$

Main technical lemma

We are now in position to state the main technical lemma of this subsection.

Lemma 4.12. *Let $\gamma \in]0, 2[$, $q > 0$ and $F : \mathbb{R}_+ \rightarrow \mathbb{R}$ be the function defined by*

$$\forall u \geq 0, \quad F(u) = \alpha u + \tilde{\alpha} \int_0^u \mathbf{1}_{[0, s]}(v) dv = \alpha u + \tilde{\alpha} u \wedge s$$

for some constants $\alpha, \tilde{\alpha} \in \mathbb{R}$ and $s > 0$. We set

$$\forall u \geq 0, \quad y_u = x_u + F(u).$$

Then, for some constant C independent of $s < r$, we have the following estimates depending on the values of the parameters $\alpha, \tilde{\alpha}$:

1) if $\alpha > 0$ and $\tilde{\alpha} = 0$

$$\mathbb{E} \left[\frac{\mathbf{1}_{\{\sup_{u \in [0, r]} y_u \in [\beta-1; \beta]\}}}{\left(\int_0^r \int_0^{2\pi} e^{\gamma y_s} \mu_Y(ds, d\sigma) \right)^q} \right] \leq C(\beta + 1) e^{(\alpha - q\gamma)\beta} r^{-3/2} e^{-\frac{\alpha^2}{2} r}.$$

2) if $\alpha = \tilde{\alpha} = 0$ then

$$\mathbb{E} \left[\frac{\mathbf{1}_{\{\sup_{u \in [0, r]} y_u \in [\beta-1; \beta]\}}}{\left(\int_0^r \int_0^{2\pi} e^{\gamma y_s} \mu_Y(ds, d\sigma) \right)^q} \right] \leq C(\beta + 1) e^{-q\gamma\beta} r^{-1/2}.$$

3) if $\alpha \geq 0$, $\tilde{\alpha} \geq 0$

$$\mathbb{E} \left[\frac{\mathbf{1}_{\{\sup_{u \in [0, r]} y_u \in [\beta-1; \beta]\}}}{\left(\int_0^r \int_0^{2\pi} e^{\gamma y_s} \mu_Y(ds, d\sigma) \right)^q} \right] \leq C e^{(\alpha + \tilde{\alpha} - q\gamma)\beta} e^{-\frac{(\alpha + \tilde{\alpha})^2}{2} s - \frac{\alpha^2}{2} (r-s)}.$$

4) if $\alpha \geq 0$, $\tilde{\alpha} < 0$ then

$$\mathbb{E} \left[\frac{\mathbf{1}_{\{\sup_{u \in [0, r]} y_u \in [\beta-1; \beta]\}}}{\left(\int_0^r \int_0^{2\pi} e^{\gamma y_s} \mu_Y(ds, d\sigma) \right)^q} \right] \leq C e^{(\alpha - q\gamma)\beta} e^{-\frac{(\alpha + \tilde{\alpha})^2 - \alpha^2 - \tilde{\alpha}^2}{2} s - \frac{\alpha^2}{2} r}.$$

Proof. Let us set $E_r = \mathbb{E} \left[\frac{\mathbf{1}_{\{\sup_{u \in [0, r]} y_u \in [\beta-1; \beta]\}}}{\left(\int_0^r \int_0^{2\pi} e^{\gamma y_s} \mu_Y(ds, d\sigma) \right)^q} \right]$. We introduce the stopping time

$$T_\beta = \inf\{s \geq 0; y_u \geq \beta - 1\}.$$

Then

$$\begin{aligned}
E_r &= \mathbb{E} \left[\frac{\mathbf{1}_{\{T_\beta < r-1\}} \mathbf{1}_{\{\sup_{u \in [0, r]} y_u \in [\beta-1; \beta]\}}}{\left(\int_0^r \int_0^{2\pi} e^{\gamma y_s} \mu_Y(ds, d\sigma) \right)^q} \right] + \mathbb{E} \left[\frac{\mathbf{1}_{\{T_\beta \geq r-1\}} \mathbf{1}_{\{\sup_{u \in [0, r]} y_u \in [\beta-1; \beta]\}}}{\left(\int_0^r \int_0^{2\pi} e^{\gamma y_s} \mu_Y(ds, d\sigma) \right)^q} \right] \\
&\leq \mathbb{E} \left[\frac{\mathbf{1}_{\{T_\beta < r-1\}} \mathbf{1}_{\{\sup_{u \in [0, r]} y_u \in [\beta-1; \beta]\}}}{e^{\gamma q y_{T_\beta}} I(T_\beta)^q} \right] + \mathbb{E} \left[\frac{\mathbf{1}_{\{T_\beta \geq r-1\}} \mathbf{1}_{\{\sup_{u \in [0, r]} y_u \in [\beta-1; \beta]\}}}{e^{\gamma q y_{r-1}} I(r-1)^q} \right] \\
&=: E_r^1 + E_r^2,
\end{aligned} \tag{4.9}$$

where we have set

$$I(a) = \int_a^{a+1} \int_0^{2\pi} e^{\gamma(y_s - y_a)} \mu_Y(ds, d\sigma).$$

We only treat E_r^1 because the same argument holds for E_r^2 . Obviously,

$$E_r^1 \leq e^{-q\gamma(\beta-1)} \mathbb{E} \left[\mathbf{1}_{\{\max_{u \in [T_\beta+1, r]} y_u - y_{T_\beta+1} \leq \beta - y_{T_\beta+1}\}} \frac{\mathbf{1}_{\{T_\beta+1 < r\}}}{I(T_\beta)^q} \right]. \quad (4.10)$$

By the strong Markov property of the Brownian motion, we see that we need the estimate

$$\mathbb{E}[I(a)^{-q} | x_{a+1} - x_a] \leq C(e^{-\gamma q(x_{a+1} - x_a)} + 1), \quad (4.11)$$

which has been proven [9, Lemma 6.1]. Denote $\mathcal{F}_t = \sigma\{x_u; u \leq t\}$. Furthermore, conditionally on \mathcal{F}_{T_β} , the random variable $\{y_{T_\beta+1} - \beta\}$ is a Gaussian random variable with variance 1 and mean $F(T_\beta + 1) - F(T_\beta)$. Let us denote by μ_β the law of the random variable $F(T_\beta + 1) - F(T_\beta)$. We deduce

$$E_r^1 \leq C e^{-q\gamma\beta} \int \int \mathbb{E} \left[\mathbf{1}_{\{\max_{u \in [T_\beta+1, r]} y_u - y_{T_\beta+1} \leq 1 - v - w\}} \right] (e^{-q\gamma v} + 1) e^{-(v-w)^2/2} dv \mu_\beta(dw).$$

Let us now observe that the random variable $F(T_\beta+1) - F(T_\beta)$ is bounded, indeed $|F(T_\beta+1) - F(T_\beta)| \leq |\alpha| + |\tilde{\alpha}|$. Let us set $c := |\alpha| + |\tilde{\alpha}|$ and $f(v) = \sup_{|w| \leq c} e^{-(v-w)^2/2}$. We obtain

$$\begin{aligned} E_r^1 &\leq C e^{-q\gamma\beta} \int \mathbb{E} \left[\mathbf{1}_{\{\max_{u \in [T_\beta+1, r]} y_u - y_{T_\beta+1} \leq -v + c + 1\}} \right] (e^{-q\gamma v} + 1) f(v) dv \\ &\leq C e^{-q\gamma\beta} \mathbb{E} \left[\mathbf{1}_{\{\max_{u \in [0, r-1]} y_u \leq \beta + \max(0, -v + 3c + 1)\}} \right] (e^{-q\gamma v} + 1) f(v) dv. \end{aligned}$$

It suffices now to combine with the various items of Lemmas 4.10 and 4.11 depending on the values of $\alpha, \tilde{\alpha}$ to complete the proof. \square

4.6 Proof of Proposition 4.1

Let us denote by $\mathcal{A}(x, r)$ the annulus with center x , inner radius $r < 1$ and outer radius 1. Let $\mathcal{A} = \mathcal{A}(x_1, |x_1 - x_2|_\epsilon)$. From (4.3) we get

$$\langle V_{\beta_1, \epsilon}(x_1) V_{\beta_2, \epsilon}(x_2) \prod_k V_{\alpha_k, \epsilon}(z_k) \rangle_\epsilon \leq C(\delta) \hat{g}(z_1)^{\Delta_{\alpha_1}} |x_1 - x_2|_\epsilon^{-\beta_1 \beta_2} I_{\mathcal{A}} \quad (4.12)$$

where $q = (\sum \alpha_i + \beta_1 + \beta_2 - 2Q)/\gamma$ and

$$I_{\mathcal{A}} = \mathbb{E} \left[\left(\int_{\mathcal{A}} \frac{1}{|z - x_1|_\epsilon^{\gamma \beta_1} |z - x_2|_\epsilon^{\gamma \beta_2}} M_\epsilon(dz) \right)^{-q} \right]. \quad (4.13)$$

We have obtained a lower bound for the expectation in (4.3) by restricting the integral (4.4) to \mathcal{A} and then used the fact that the rest of the integrand in (4.4) is bounded from below by δ dependent constant, uniformly in ϵ if $z \in \mathcal{A}$. Furthermore, for $z \in \mathcal{A}$, we have $|z - x_2| \leq 2|z - x_1|$ so that

$$I_{\mathcal{A}} \leq C \mathbb{E} \left[\left(\int_{\mathcal{A}} \frac{1}{|z - x_1|_\epsilon^{\gamma(\beta_1 + \beta_2)}} M_\epsilon(dz) \right)^{-q} \right]. \quad (4.14)$$

It is convenient at this point to change the regularization. Let M'_ϵ be the chaos measure defined with the circle average regularization field X'_ϵ . We have

$$\mathbb{E} X'_\epsilon(u) X'_\epsilon(v) \leq C + X_\epsilon(u) X_\epsilon(v)$$

where C is uniform in ϵ . Hence by Kahane convexity (4.14) holds for this regularization too (by a different constant C).

The expectation in the right-hand side of (4.14) may be now written in terms of the radial decomposition of the chaos measure explained in Section 4.5:

$$\mathbb{E}\left[\left(\int_{\mathcal{A}} \frac{1}{|z - x_1|_{\epsilon}^{\gamma(\beta_1 + \beta_2)}} M'_{\epsilon}(dz)\right)^{-q}\right] = \mathbb{E}\left[\left(\int_0^{-\ln|x_1 - x_2|_{\epsilon}} \int_0^{2\pi} e^{\gamma y_s} \mu_{Y_{\epsilon}}(ds, d\sigma)\right)^{-q}\right]$$

where Y_{ϵ} is the circle average of Y and

$$y_s = x_s + (\beta_1 + \beta_2 - Q)s.$$

Now we partition the probability space according to the values of the maximum of y_s . Define

$$M_n = \left\{ \max_{s \in [0, -\ln|x_1 - x_2|_{\epsilon}]} y_s \in [n - 1, n] \right\}, \quad n \geq 1, \quad (4.15)$$

$$M_0 = \left\{ \max_{s \in [0, -\ln|x_1 - x_2|_{\epsilon}]} y_s \leq 0 \right\}. \quad (4.16)$$

Then

$$I_{\mathcal{A}} \leq C \sum_{n \geq 0} \mathbb{E}\left[\mathbf{1}_{M_n} \left(\int_0^{-\ln|x_1 - x_2|_{\epsilon}} \int_0^{2\pi} e^{\gamma y_s} \mu_{Y_{\epsilon}}(ds, d\sigma)\right)^{-q}\right] := C \sum_{n \geq 0} A_n. \quad (4.17)$$

By Lemma 4.12 item 1, with $\alpha = \beta_1 + \beta_2 - Q > 0$, $\tilde{\alpha} = 0$, $q = \frac{\beta_1 + \beta_2 + \sum_i \alpha_i - 2Q}{\gamma}$ and $\beta = n$, we deduce

$$A_n \leq C(n+1)e^{-(\sum_i \alpha_i - Q)n} |\ln|x_1 - x_2|_{\epsilon}|^{-3/2} |x_1 - x_2|_{\epsilon}^{\frac{(\beta_1 + \beta_2 - Q)^2}{2}}.$$

Combining this with (4.17) and (4.12) we arrive at the claim since $\frac{(\beta_1 + \beta_2 - Q)^2}{2} - \beta_1 \beta_2 = 2(\Delta_{(\beta_1 + \beta_2) \wedge Q} - \Delta_{\beta_1} - \Delta_{\beta_2})$ if $\beta_1 + \beta_2 > Q$.

For $\beta_1 + \beta_2 = Q$ we use Lemma 4.12 2) and for $\beta_1 + \beta_2 < Q$ $I_{\mathcal{A}}$ is uniformly bounded in ϵ . \square

4.7 Proof of Proposition 4.3

We proceed as in the previous section, starting with

$$\langle V_{\beta_1, \epsilon}(x_1) V_{\beta_2, \epsilon}(x_2) V_{\beta_3, \epsilon}(x_3) \prod_k V_{\alpha_k, \epsilon}(z_k) \rangle_{\epsilon} \leq C(\delta) |x_1 - x_2|_{\epsilon}^{-\beta_1 \beta_2} |x_1 - x_3|_{\epsilon}^{-\beta_1 \beta_3} |x_2 - x_3|_{\epsilon}^{-\beta_2 \beta_3} I_{\mathcal{A}} \quad (4.18)$$

where

$$q = (\sum \alpha_i + \beta_1 + \beta_2 + \beta_3 - 2Q)/\gamma \quad (4.19)$$

and

$$I_{\mathcal{A}} = \mathbb{E}\left[\left(\int_{\mathcal{A}} \frac{1}{|z - x_1|_{\epsilon}^{\gamma \beta_1} |z - x_2|_{\epsilon}^{\gamma \beta_2} |z - x_3|_{\epsilon}^{\gamma \beta_3}} M_{\epsilon}(dz)\right)^{-q}\right]. \quad (4.20)$$

The choice of the region \mathcal{A} will depend on the β_i 's. We need to consider several cases.

1. $\beta_1 + \beta_2 \geq Q$, $\beta_3 \geq 0$. We take $\mathcal{A} = \mathcal{A}(x_1, |x_1 - x_2|_{\epsilon})$. Inserting the estimates, valid for $z \in \mathcal{A}$,

$$|z - x_2|_{\epsilon} \leq 2|z - x_1|_{\epsilon}, \quad |z - x_3|_{\epsilon} \leq 2|x_1 - x_3|_{\epsilon} \mathbf{1}_{\{|z - x_1|_{\epsilon} \leq |x_1 - x_3|_{\epsilon}\}} + 2|z - x_1|_{\epsilon} \mathbf{1}_{\{|z - x_1|_{\epsilon} > |x_1 - x_3|_{\epsilon}\}} \quad (4.21)$$

into (4.18) and then use the polar decomposition of the chaos measure around 0 we deduce that the expectation in the right-hand side of (4.18) is bounded by

$$I_{\mathcal{A}} \leq C \sum_{n \geq 0} A_n \quad (4.22)$$

$$A_n = \mathbb{E} \left[\mathbf{1}_{M_n} \left(\int_0^{-\ln |x_1 - x_2|_\epsilon} \int_0^{2\pi} e^{\gamma y_s} \mu_{Y_\epsilon}(ds, d\sigma) \right)^{-q} \right] \quad (4.23)$$

where y_s is the process

$$y_s = x_s + (\beta_1 + \beta_2 - Q)s + \beta_3 s \wedge \ln |x_1 - x_3|_\epsilon^{-1} \quad (4.24)$$

and M_n is as in (4.15) and (4.16).

We can now apply item 3 of Lemma 4.12 with $\alpha = \beta_1 + \beta_2 - Q$, $\tilde{\alpha} = \beta_3$, $\beta = n$, $r = \ln |x_1 - x_2|_\epsilon^{-1}$ and $s = \ln |x_1 - x_3|_\epsilon^{-1}$ and q as in (4.19) to bound

$$A_n \leq C e^{-n(\sum_i \alpha_i - Q)} |x_1 - x_2|_\epsilon^{\frac{(\beta_1 + \beta_2 - Q)^2}{2}} |x_1 - x_3|_\epsilon^{\frac{(\beta_1 + \beta_2 + \beta_3 - Q)^2}{2} - \frac{(\beta_1 + \beta_2 - Q)^2}{2}}. \quad (4.25)$$

Combining (4.18)+(4.22)+(4.25) with $|x_2 - x_3| \geq |x_1 - x_3|$ the claim follows.

2. $\sum_i \alpha_i + \beta_3 - Q > 0$, $\beta_1 + \beta_2 \geq Q$, $\beta_3 < 0$. We replace \mathcal{A} in the previous case by the half annulus $\mathcal{A}' := \mathcal{A} \cap \{z : \arg(\frac{z - x_1}{x_3 - x_1}) \in [\frac{\pi}{2}, \frac{3\pi}{2}]\}$. This has the virtue that for $z \in \mathcal{A}'$

$$C|z - x_3|_\epsilon \geq |x_1 - x_3|_\epsilon \mathbf{1}_{\{|z - x_1|_\epsilon \leq |x_1 - x_3|_\epsilon\}} + |z - x_1|_\epsilon \mathbf{1}_{\{|z - x_1|_\epsilon > |x_1 - x_3|_\epsilon\}}.$$

Then we apply Lemma 4.12 item 4 since $\tilde{\alpha} = \beta_3 < 0$ to get

$$A_n \leq C e^{-n(\sum_i \alpha_i + \beta_3 - Q)} |x_1 - x_2|_\epsilon^{\frac{(\beta_1 + \beta_2 - Q)^2}{2}} |x_1 - x_3|_\epsilon^{(\beta_1 + \beta_2 - Q)\beta_3}. \quad (4.26)$$

The claim follows by using $|x_1 - x_3| \leq |x_2 - x_3|$ and $(\beta_2 - Q)\beta_3 \geq 0$.

3. $\sum_i \alpha_i > Q$, $\beta_1 + \beta_2 < Q$, $\beta_2 > 0$ and $\beta_1 + \beta_2 + \beta_3 \geq Q$. Now we take $\mathcal{A} = \mathcal{A}(x_3, |x_3 - x_1|)$. On \mathcal{A} the integrand in (4.20) is bounded by $|z - x_3|^{-\gamma(\beta_1 + \beta_2 + \beta_3)}$ and we arrive at

$$A_n \leq C e^{-n(\sum_i \alpha_i - Q)} |x_1 - x_3|_\epsilon^{\frac{(\beta_1 + \beta_2 + \beta_3 - Q)^2}{2}}.$$

4. $\sum_i \alpha_i > Q$, $\beta_1 + \beta_2 < Q$, $\beta_2 < 0$ and $\beta_1 + \beta_2 + \beta_3 \geq Q$. By taking \mathcal{A} an appropriate half of the annulus in **3.** we can guarantee that $C|z - x_2| > |z - x_3|$. Then we may repeat the argument in **3.**

5. $\beta_1 + \beta_2 < Q$, $\beta_1 + \beta_2 + \beta_3 < Q$. Take \mathcal{A} unit ball with distance 1 to $\{x_1, x_2, x_3\}$. Then $I_{\mathcal{A}} \leq C$ and we get from (4.18) the desired bound since $(\beta_1 + \beta_2)\beta_3 = 2(\Delta_{\beta_1 + \beta_2} - \Delta_{\beta_1} - \Delta_{\beta_2})$.

We have listed all the possibilities so that the proof of Proposition 4.3 is complete. \square

4.8 Proof of Proposition 4.7

Without loss let $y = 0$ and $|z_i| \geq 2$. From (4.3) we obtain

$$\mathcal{F}_\epsilon(0, z) = C e^{-\frac{\gamma^2}{2} G_\epsilon(z)} e^{-\frac{\gamma}{2} \sum_i \alpha_i G_\epsilon(z - z_i)} h_\epsilon(z) \quad (4.27)$$

with

$$h_\epsilon(z) = \mathbb{E} \left[\int e^{-\frac{\gamma^2}{2} G_\epsilon(u - z) + \gamma^2 G_\epsilon(u)} \rho_\epsilon(u) dM_\epsilon(u) \right]^{-s}$$

Here $s = \sum \alpha_i + \frac{1}{2}\gamma - 2Q$ and $\rho_\epsilon(z) \leq C$ on $|z| \leq 1$. Since $h_\epsilon(z) \leq C$ and

$$C^{-1}|z|_\epsilon \leq e^{-G_\epsilon(z)} \leq C|z|_\epsilon$$

we only need to consider the case $|z| \leq \epsilon$. Set $k_\epsilon(z, u) = \partial_z e^{-\frac{\gamma^2}{2}G_\epsilon(u-z) + \gamma^2 G_\epsilon(u)}$. We compute

$$\begin{aligned} \partial_z h_\epsilon(z) &= \int_{|u| \leq 1} k_\epsilon(z, u) \rho(u) \mathbb{E} \left[\int e^{-\frac{\gamma^2}{2}G_\epsilon(v-z) + \gamma^2 G_\epsilon(v) + \gamma^2 G_\epsilon(v-u)} \rho(v) dM_\epsilon(v) \right]^{-s-1} du \\ &\quad + \mathbb{E} \int_{|u| \geq 1} du k_\epsilon(z, u) \rho(u) \left[\int e^{-\frac{\gamma^2}{2}G_\epsilon(v-z) + \gamma^2 G_\epsilon(v)} \rho(v) dM_\epsilon(v) \right]^{-s-1} dM_\epsilon(u) \end{aligned} \quad (4.28)$$

where in the first term we got rid of $dM_\epsilon(u)$ by Girsanov transform. Now use

$$|\partial_z G_\epsilon(u-z)| \leq C|u-z|_\epsilon^{-1}$$

whereby

$$|k_\epsilon(z, u)| \leq C e^{-\frac{\gamma^2}{2}G_\epsilon(u-z) + \gamma^2 G_\epsilon(u)}$$

for $|u| \geq 1$, $|z| \leq \epsilon$ so that the second term in (4.28) is bounded by

$$\mathbb{E} \left[\int_{|u| \geq 1} e^{-\frac{\gamma^2}{2}G_\epsilon(v-z) + \gamma^2 G_\epsilon(v)} \rho(v) dM_\epsilon(v) \right]^{-s} \leq C.$$

In the first term the expectation is bounded by C and

$$|k_\epsilon(z, u)| \leq C|u|_\epsilon^{-\frac{\gamma^2}{2}-1}$$

since $|z| \leq \epsilon$. We obtain then

$$|\partial_z h_\epsilon(z)| \leq C(\epsilon^{1-\frac{\gamma^2}{2}} + 1).$$

The second term in (4.27) satisfies

$$|\partial_z e^{-\frac{\gamma^2}{2} \sum_i \alpha_i G_\epsilon(z-z_i)}| \leq C$$

and the first term

$$|\partial_z e^{-\frac{\gamma^2}{2}G_\epsilon(z)}| \leq C\epsilon^{\frac{\gamma^2}{2}-1}.$$

Altogether we get for $|z| \leq \epsilon$

$$|\mathcal{F}_\epsilon(0, z) - \mathcal{F}_\epsilon(0, 0)| \leq C\epsilon^{\frac{\gamma^2}{2}-1}|z|$$

□

4.9 Proof of Proposition 4.5

From (3.27) we get

$$\langle V_{\gamma, \epsilon}(y) \prod_k V_{\alpha_k, \epsilon}(z_k) \rangle_\epsilon \leq C(\delta) |y|^{-\gamma \sum \alpha_i} f(y)$$

where

$$f(y) = \mathbb{E} \left[\int_{A_\delta} e^{\gamma X(x) - \frac{\gamma^2}{2} \mathbb{E}[X(x)^2]} \frac{1}{|x-y|^{\gamma^2}} \prod_l \frac{1}{|x-z_l|^{\gamma \alpha_l}} \hat{g}(x)^{1-\frac{\gamma}{4} \sum_l \alpha_l} dx \right]^{-\frac{\sum \alpha_i + \gamma - 2Q}{\gamma}}$$

where A_δ is the annulus around origin with radi $2/\delta$ and $3/\delta$. We get

$$\lim_{y \rightarrow \infty} |y|^{-\gamma(\sum \alpha_i + \gamma - 2Q)} f(y) < \infty$$

so that

$$\langle V_{\gamma, \epsilon}(y) \prod_k V_{\alpha_k, \epsilon}(z_k) \rangle_\epsilon \leq C(\delta) |y|^{\gamma^2 - 2Q\gamma} = C(\delta) |y|^{-4}$$

□

4.10 Proof of Proposition 4.6

Let us assume that all the insertion points z_1, \dots, z_N are all distinct from 0 (otherwise replace the Möbius transform $z \mapsto 1/z$ in what follows by $z \mapsto 1/(z - a)$ for some a distinct from all the $(z_i)_i$).

By Kahane convexity we have

$$\langle V_{\beta_1, \epsilon}(x_1) V_{\beta_2, \epsilon}(x_2) \prod_i V_{\alpha_i, \epsilon}(z_i) \rangle_\epsilon \leq C \langle V_{\beta_1, \epsilon}(x_1) V_{\beta_2, \epsilon}(x_2) \prod_i V_{\alpha_i, \epsilon}(z_i) \rangle. \quad (4.29)$$

for some irrelevant constant C (which may change along lines). Möbius invariance of the Liouville field ϕ (see [8, section 3.2]) gives

$$\langle V_{\beta_1, \epsilon}(x_1) V_{\beta_2, \epsilon}(x_2) \prod_i V_{\alpha_i, \epsilon}(z_i) \rangle = \langle V_{\beta_1, \epsilon}^\psi(x_1) V_{\beta_2, \epsilon}^\psi(x_2) \prod_i V_{\alpha_i, \epsilon}^\psi(z_i) \rangle$$

where

$$V_{\beta, \epsilon}^\psi(x) = (A\epsilon)^{\frac{\beta^2}{2}} e^{\beta((X \circ \psi)_\epsilon(x) + \frac{1}{2}Q(\ln \hat{g} \circ \psi)_\epsilon(x) + Q(\ln |\psi'|)_\epsilon + c)}.$$

As in (3.26) we get

$$(4.29) \leq C \left(\prod_i e^{2\Delta_{\beta_i}(\ln |\psi'|)_\epsilon(x_i)} \right) e^{\frac{1}{2} \sum_{i,j} \beta_i \beta_j G_\epsilon(x_i, x_j)} \quad (4.30)$$

$$\mathbb{E} \left[\left(\int_{\mathbb{C}} e^{\gamma X(x) - \frac{\gamma^2}{2} \mathbb{E}[X(x)^2] + \gamma \sum_i \beta_i G_\epsilon^\psi(x, x_i)} \hat{g}(x) dx \right)^{-s} \right] \quad (4.31)$$

where $\{\beta_1, \dots, \beta_{n+2}\} = \{\beta_1, \beta_2, \alpha_1, \dots, \alpha_n\}$, $\{x_1, \dots, x_{n+2}\} = \{y, x, z_1, \dots, z_n\}$ and

$$G_\epsilon^\psi(x, y) = \int G(x, v) \rho_\epsilon(y - \frac{1}{v}) \frac{dv}{|v|^4}.$$

Note that

$$\lim_{\epsilon \rightarrow 0} G_\epsilon^\psi(x, y) = G(x, \frac{1}{y}) \quad x \neq \frac{1}{y}.$$

If $x, 1/y$ belong to a ball $B(0, R)$ for some $R > 0$ we have the estimate

$$G_\epsilon^\psi(x, y) \geq \ln \frac{1}{|x - \frac{1}{y}| + \epsilon R^2} - C_R \quad (4.32)$$

Indeed, by applying Jensen to $-\ln$ and the triangle inequality, we get

$$G_\epsilon^\psi(x, y) = \int -\ln |x - \frac{1}{y + \epsilon u}| \rho(|u|^2) du \geq -\ln \left(|x - \frac{1}{y}| + \int \left| \frac{1}{y} - \frac{1}{y + \epsilon u} \right| \rho(|u|^2) du \right).$$

Now, notice that $\int \left| \frac{1}{y} - \frac{1}{y + \epsilon u} \right| \rho(|u|^2) du = \frac{\epsilon}{y} \int \left| \frac{u}{y + \epsilon u} \right| \rho(|u|^2) du \leq C\epsilon R^2$.

Now we focus on the item 1 (items 2 and 3 are dealt with in the same way). In that case $\beta_1, \beta_2 > 0$. The expectation in (4.30) is then less than

$$C_\delta \mathbb{E} \left[\left(\int_{\mathcal{A}} \left| z - \frac{1}{x_1} \right|^{-\gamma \beta_1} \left| z - \frac{1}{x_2} \right|^{-\gamma \beta_2} M_\gamma(dx) \right)^{-s} \right]$$

where the set \mathcal{A} is given by $\mathcal{A} := \{z \in \mathbb{C}; |\frac{1}{x_2} - \frac{1}{x_1}|_\epsilon \leq |z - \frac{1}{x_1}| \leq 1\}$. We can then complete the proof as done in the proof of Proposition 4.1. \square

5 Appendix

5.1 Reminder on integral calculus and singular transforms in the complex plane

The Cauchy transform is defined by the following limit if $f(y)$ is some function in $L^1(dy)$

$$\mathcal{C}(f)(z) = \lim_{\epsilon \rightarrow 0} \frac{1}{\pi} \int \frac{1_{|z-y|>\epsilon}}{(z-y)} f(y) dy,$$

where the above convergence holds in $L^1_{loc}(dz)$. It is well known that the following relation holds in the sense of distributions

$$\partial_{\bar{z}} \mathcal{C}(f)(z) = f(z) \quad (5.1)$$

The Beurling transform of a function $f \in L^p(dy)$ for $p > 1$ is defined by the following limit

$$\mathcal{B}(f)(z) := -\lim_{\epsilon \rightarrow 0} \frac{1}{\pi} \int \frac{1_{|z-y|>\epsilon}}{(z-y)^2} f(y) dy \quad (5.2)$$

Therefore, the Beurling transform is a singular Calderon-Sygmund type operator and the above convergence holds in $L^p(dz)$. It is well known that the transform is continuous in L^p (see [1] for example), i.e. there exists some constant C_p such that for all f in $L^p(dz)$

$$|\mathcal{B}(f)|_{L^p} \leq C_p |f|_{L^p}. \quad (5.3)$$

Let us mention that it is still an open problem to determine the optimal C_p . In our proofs, we will need a slightly stronger convergence in (5.2) when f is Hölder (the proof is simple and left to the reader):

Lemma 5.1. *Let f be some function in $L^p(dy)$ for $p > 1$. Let D be some open domain such that f is Hölder in the domain D , i.e. there exists some constant $C > 0$ and $\alpha > 0$ such that*

$$|f(y) - f(x)| \leq C|y - x|^\alpha.$$

Then, the convergence in (5.2) holds in the space of continuous functions. Also, if $\chi(y)$ is an isotropic smooth function with support in $B(0, 1)$ and worth 1 inside $B(0, \frac{1}{2})$ or the indicator $1_{|y| \leq 1}$ then we have the following expression for the Beurling transform in D

$$\mathcal{B}(f)(z) = \int \frac{f(y) - \chi(z-y)f(z)}{(z-y)^2} dy$$

Now, we turn to the relation between the Cauchy transform and the Beurling transform: it is well known that for all functions $f \in L^p(dy)$ for $p > 1$, the following holds in the sense of distributions

$$\partial_{\bar{z}} \mathcal{C}(f)(z) = \mathcal{B}(f)(z) \quad (5.4)$$

In particular, by taking the $\partial_{\bar{z}}$ derivative of this relation and using (5.4), we deduce that the following holds in the sense of distributions:

$$\partial_{\bar{z}} \mathcal{B}(f)(z) = \partial_{\bar{z}} f(z) \quad (5.5)$$

5.2 Analysis of the hypergeometric PDE

Here is a lemma that will be useful in the study of the four point correlation function:

Lemma 5.2. *The real solutions in $\mathbb{C} \setminus \{0, 1\}$ to the equation*

$$\partial_{zz}^2 u(z) + \frac{(c - z(a + b + 1))}{z(1-z)} \partial_z u(z) - \frac{ab}{z(1-z)} u(z) = 0 \quad (5.6)$$

is a 4d vector space spanned by $|F_-(z)|^2$, $\text{Re}(F_-(z)F_+(z))$, $\text{Im}(F_-(z)F_+(z))$, $|F_+(z)|^2$.

Proof. First, it is clear that $|F_-(z)|^2, \operatorname{Re}(F_-(z)F_+(z)), \operatorname{Im}(F_-(z)F_+(z)), |F_+(z)|^2$ are four linearly independent solutions to (5.6) (for instance, one can look at the expansion around $z = 0$).

Now, we want to show that the solutions of the equation (5.6) is a vector space of dimension less or equal to 4. Therefore, we consider a real solution u to the equation

$$\partial_{zz}^2 u(z) + \frac{(c - z(a + b + 1))}{z(1 - z)} \partial_z u(z) - \frac{ab}{z(1 - z)} u(z) = 0 \quad (5.7)$$

If we take the $\partial_{\bar{z}\bar{z}}$ derivative of (5.6) then we see that u is a solution of a PDE whose highest degree operator is Δ^2 hence of a analytic hypoelliptic system. Therefore, in order to show that solutions coincide it is enough to show that they coincide on an open subset of $\mathbb{C} \setminus \{0, 1\}$.

If we take the real part of equation (5.7), we get:

$$\frac{\partial^2 u}{\partial x^2}(x, y) - \frac{\partial^2 u}{\partial y^2}(x, y) + G_1(x, y) \frac{\partial u}{\partial x}(x, y) + G_2(x, y) \frac{\partial u}{\partial y}(x, y) + G_3(x, y) u(x, y) = 0 \quad (5.8)$$

This first hyperbolic equation is determined up to $u_0(x) = u(x, 0)$ and $v_0(x) = \frac{\partial}{\partial y} u(x, 0)$. We want to determine u_0 and v_0 . Now, we take the imaginary part of (5.7); using the fact that a, b, c are real, this yields

$$-\frac{1}{2} v_0'(x) + \frac{(c - x(a + b + 1))}{x(1 - x)} v_0(x) = 0$$

Hence v_0 lives in a space of dimension 1. We know want to show that given the data v_0 , the function u_0 lives at most in a space of dimension 3. Recall now that taking the imaginary part of (5.6) yields an equation of the form

$$\frac{\partial^2 u}{\partial xy}(x, y) + \bar{G}_1(x, y) \frac{\partial u}{\partial x}(x, y) + \bar{G}_2(x, y) \frac{\partial u}{\partial y}(x, y) + \bar{G}_3(x, y) u(x, y) = 0 \quad (5.9)$$

If we differentiate this equation with respect to y , we get something of the form

$$\frac{\partial}{\partial x} \left(\frac{\partial^2 u}{\partial y^2} \right)(x, y) + \widetilde{G}_1(x, y) \frac{\partial u}{\partial x}(x, y) + \widetilde{G}_2(x, y) \frac{\partial u}{\partial y}(x, y) + \widetilde{G}_3(x, y) u(x, y) + \widetilde{H}_1(x, y) \frac{\partial^2 u}{\partial xy}(x, y) + \widetilde{H}_2(x, y) \frac{\partial^2 u}{\partial y^2}(x, y) = 0$$

Now, if we plug in the expression (5.8) of $\frac{\partial^2 u}{\partial y^2}$ and the expression (5.9) for $\frac{\partial^2 u}{\partial xy}(x, y)$ and put $y = 0$, we get an equation of the form

$$u_0^{(3)}(x) + f_1(x) u_0^{(2)}(x) + f_2(x) u_0'(x) + f_3(x) u_0(x) = f_4(x)$$

where the function $f_4(x)$ is a linear combination of $v_0(x)$ and its derivatives (the coefficients in the combination are functions of x). Hence the vector space of the above equation is 3. Combining everything, we get that the set of solutions is at most of dimension 4. \square

5.3 Useful formulas and special functions

Here, we recall usual formulas on the hypergeometric functions and the Γ function. We have

$$\Gamma(z)\Gamma(1 - z) = \frac{\pi}{\sin \pi z} \quad (5.10)$$

We set the following definition in the paper

$$l(x) = \frac{\Gamma(x)}{\Gamma(1 - x)}$$

We have

$$l(1 - x)l(x) = 1, \quad l(x)l(-x) = -x^2, \quad l(1 + x) = -x^2 l(x)$$

Now, according to formula (1.3) and the formula for A_1 page 504 in [16], we have for $\alpha, \beta > 0$ and $\alpha + \beta < 1$

$$\int_{\mathbb{R}^2} |z|^{2(\alpha-1)} |z-1|^{2(\beta-1)} dz = \left(\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \right)^2 \frac{\sin(\alpha\pi)\sin(\beta\pi)}{\sin((\alpha+\beta)\pi)}$$

By using (5.10), we get

$$\int_{\mathbb{R}^2} |z|^{2(\alpha-1)} |z-1|^{2(\beta-1)} dz = \pi \frac{l(\alpha)l(\beta)}{l(\alpha+\beta)} = \pi \frac{1}{l(1-\alpha)l(1-\beta)l(\alpha+\beta)}. \quad (5.11)$$

This formula can be analytically continued to get for $\alpha, \beta > 0$ and $1 < \alpha + \beta < 3/2$

$$\int_{\mathbb{R}^2} |z|^{2(\alpha-1)} (|z-1|^{2(\beta-1)} - |z|^{2(\beta-1)}) dz = \pi \frac{1}{l(1-\alpha)l(1-\beta)l(\alpha+\beta)}. \quad (5.12)$$

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